Mathematics of Games
(Continuation Report)

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Contents

Chapter 1

Introduction

This introduction describes the historical and mathematical background to our research.

1.1 Games

We are interested in the use of mathematical structures called *games* to give denotational models of programming languages and logics. We concentrate on programming languages that extend the simply-typed λ-calculus, and on subsystems of Girard's linear logic (Girard, 1987).

A game is a set of constraints that govern dialogue between two opposing entities, whom we term Proponent and Opponent. (The choice of names makes it clear whose side we are on.) Each player decides upon a *strategy* that will dictate how he should respond at each point in the game, and the eventual outcome of the game depends only on the strategies used by the players.

The study of games has a long prehistory in mathematical logic. In the 1950s, Paul Lorenzen used a notion of dialogue game to describe the semantics of intuitionistic logic; and games of one sort or another have also been used in model theory (by Ehrenfeucht and Fraïssé) and set theory (the Axiom of Determinacy).

The modern branch of the subject began with Blass's game semantics for linear logic (Blass, 1992), which draws on some much earlier work of his (Blass, 1972). Blass's games do not quite form a category, since his composition operation is not always associative. This defect was repaired fairly rapidly in subsequent work on linear logic (Abramsky and Jagadeesan, 1992; Hyland and Ong, 1993), which marks the beginning of the use of *categories of games* to give semantic interpretations for logic. (Though these are not the first categories of games: that honour goes to the category of Conway games (Conway, 1976) described by Joyal (1977)).

This work on linear logic was rapidly followed by game-semantic interpretations of programming languages. The first syntax-free descriptions of the fully abstract model of Scott–Plotkin's language PCF were discovered by Abramsky et al. (2000) and Hyland and Ong (2000), working independently. Since that time, games have been used to model a wide variety of different programming languages. An especially noteworthy achievement is the description of a fully abstract model for Reynolds's *Idealized Algol* by Abramsky and McCusker (2000, 1999).

The importance of games rests on the fact that they provide an intensional analysis of computation, because their morphisms represent algorithms rather than just extensional functions. So there is a sense in which game semantics occupies a middle ground between traditional operational and denotational semantics. From this angle, an important precursor is the work of Berry and Curien on sequential algorithms (Berry and Curien, 1982; Curien, 1994); and in fact the sequential algorithms model is equivalent to a category of games, as shown by Lamarche (1992) and Curien (1993).

It turns out that he extensional elements of the sequential algorithms model are precisely the *strongly stable functions* (Ehrhard, 1999). Longley (2002) has made a persuasive case that the strongly stable functions constitute a canonical class of sequentially computable higher-order functions. Moreover the category of hypercoherences (Ehrhard, 1993) is a compellingly simple linearisation of the strongly stable model. So there is a definite connection between games and hypercoherences, whose precise nature remains unclear. (Melliès (2003) has made some progress unravelling it, but much remains to be understood.)

1.2 The quest

The fundamental ideas of game semantics are extremely concrete and simple. A game is a set of plays; a play is a sequence of moves, alternating between Opponent moves and Proponent moves; and a move is an indivisible atom, an arbitrary symbol. The simplicity and naturality of these basic definitions has surely contributed to the popular appeal of game semantics.

But as we define the mathematical structure of the category of games, difficulties begin to mount. The definition of a morphism between two games is not too hard; the operation of composing two morphisms is complicated, but manageably so. The first real difficulty comes when trying to prove that composition is associative, which demands a thicket of elementary manipulations. Arguably this is not so bad, since the basic structure of the category need only be established once and is then free for the using. Perhaps the real problem is that there are a large number of variations of the basic notion of game that give rise to well-behaved and useful categories; and an even larger number that do not. It is not at all clear which modifications of the basic definitions will 'work' and which not.

Beneath these practical difficulties lies a conceptual one. It is not clear *why* certain elementary definitions should give rise to a richly structured category, while others – such as Blass's original games for linear logic – do not give a category at all.

The quest, then, is for a more conceptual approach to games, that allows us to abstract away from the nitty-gritty and see the essential structures clearly.

Also the relationship between intensional game models and intensional (e.g. domain) models is far from being clearly understood, and it would be nice to clarify the connections here. We tend to favour approaches that offer a handle on this problem.

1.3 Summary of results

We began by studying the category Gam of *simple games* (Hyland, 1997). The objects of this category are games, and a map $G_1 \to G_2$ is a strategy on a compound game $G_1 \multimap G_2$ that combines G_1 and G_2. As mentioned above, the first unexpectedly difficult problem here is to show the associativity of composition; so we were looking particularly for an alternative (but equivalent) way to define the notion of strategy so that composition would obviously be associative.

It had earlier been observed by Andrea Schalk that there is a faithful functor P from Gam to the category Rel of sets and relations.

This functor maps a game to the set of its positions, and so a morphism $G_1 \to G_2$ is mapped to a relation between $P(G_1)$ (i.e. the set of positions of G_1) and $P(G_2)$. This is a definite step in the right direction, because the composite of two strategies can be calculated by representing them as relations and then composing the resulting relations; and moreover, composition of relations is clearly associative.

Therefore one could define a strategy to be a relation of a particular kind, and the basic categorical structure would immediately follow. The sticking point is indicated by the words 'of a particular kind' above: the functor R is not full, so not *every* relation between $P(G_1)$ and $P(G_2)$ actually represents a strategy. The challenge is to find a reasonable characterisation of the 'strategic' relations – the ones that do in fact represent strategies.

Our solution to this problem is described in Chapter 2. An interesting consequence of our characterisation is that there is a natural way to give each set $P(G)$ the structure of a hypergraph (Melliès, 2003) in such a way that every strategic relation preserves the hypergraph structure. It follows that the functor P can be factorised as

$$\begin{tikzcd}\mathrm{Gam} \arrow[r, "H"] & \mathrm{HGraph} \arrow[r, "U"] & \mathrm{Rel}\end{tikzcd}$$

where HGraph is Melliès's category of hypergraphs, and U is the usual forgetful functor from hypergraphs to relations.

Just as the category of hypercoherences linearises the strongly stable model, so too does the very similar category of hypergraphs. Thus our functor H provides a solid bridge between the intensional and extensional worlds. We hope that it will be a useful starting point for more profound investigations.

Having reduced games to hypergraphs (almost), it is natural to wonder about the category of hypergraphs itself. The category of hypercoherences is highly structured; in fact it has enough structure to model the whole of classical linear logic; a hypergraph is a hypercoherence whose atoms are polarised in the sense that each of them occurs either 'positively' or 'negatively', and the definitions of the various connectives are modified to take this polarisation into account in a natural way. Thus the category of hypergraphs still has all the structure needed to model linear logic. The puzzle is *why* this should be possible; to put it crudely, you don't get that much structure just by chance! Although there are several known abstract constructions that produce polarisation effects (Hyland and Schalk, 2003, §3), the category of hypergraphs does not seem to arise from any of them.

Thus we developed a theory of partial monoidal categories, which allows us to describe the precise formal relationship between hypercoherences and hypergraphs. In summary:

1. There is a categorical completion process HC – generalising the coherence completion of Hu and Joyal (1999) – such that the category of hypercoherences is HC($\mathbf{1}$), where $\mathbf{1}$ is the terminal category.

   In fact for any category $\mathbb{C}$, the completion HC($\mathbb{C}$) contains the Hu-Joyal coherence completion Coh($\mathbb{C}$) as a full subcategory (and this in turn has a full subcategory CCoh($\mathbb{C}$) that is the free completion of $\mathbb{C}$ with products, coproducts and a zero object).

2. This process has the effect that partial monoidal structure on a category $\mathbb{C}$ induces ordinary (total) monoidal structure on HC($\mathbb{C}$). For example if $\mathbb{C}$ is partial symmetric monoidal closed then HC($\mathbb{C}$) is symmetric monoidal closed.

   This observation is the crucial one, and even works for the ordinary coherence completion. Something similar but much more subtle is also true about linear exponentials: if we define carefully enough what it means for a linear category to have a 'partial linear exponential', then any partial linear exponential on $\mathbb{C}$ induces a linear exponential on HC($\mathbb{C}$).

3. If the category $\mathbb{D}$ is symmetric monoidal then $\mathbb{D}+\mathbb{D}^{\mathrm{op}}$ is partial $*$-autonomous.

   The linear exponential is again more subtle. If $\mathbb{D}$ has a linear exponential comonad and also a monad satisfying certain conditions, we can combine them to produce a partial linear exponential on $\mathbb{D}+\mathbb{D}^{\mathrm{op}}$.

The category of hypergraphs is HC($\mathbf{1}+\mathbf{1}^{\mathrm{op}}$), using the trivial symmetric monoidal closed structure on $\mathbf{1}$ to derive a partial $*$-autonomous structure on $\mathbf{1}+\mathbf{1}^{\mathrm{op}}$, which in turn produces the usual $*$-autonomous structure of hypergraphs. For the exponential we start with the trivial monad and comonad.

This theory is described in Chapter 3. (The presentation there is somewhat unsatisfactory, and would certainly benefit from a stronger dose of higher-dimensional algebra. But it will require a non-trivial effort to find a good presentation at the right level of generality.)

Two significant advances have been made recently by other researchers (Cockett and Seely, 2004; Hyland, 2004). The work of Cockett and Seely implies that the category of finite simple games with winning strategies is the initial model of a certain categorical situation. The result in this form is quite special; what's exciting is that similar characterisations appear to exist for many other categories of games, including the important arena games (with innocent strategies) defined by Hyland and Ong (2000). Moreover it is possible (even easy) to deduce the existence of the functor H from this characterisation (at least in the case of finite games). This is discussed in more detail in Chapter 4.

Hyland's work offers a powerful abstraction of the notion of innocent strategy. His construction is sketched in Chapter 4 too. It is not yet clear how it relates to our approach.

Finally we describe our plans for the next two years.

Chapter 2

Strategies as Relations

2.1 Introduction

In the category of simple games (Hyland, 1997), a linear morphism $f : G \to G'$ is a strategy for the compound game $G \multimap G'$. The composite of two morphisms is then defined by the method of 'parallel composition + hiding'.[1] This operation is rather complicated and difficult to use: it is notoriously tedious, for example, to prove the simple fact that composition is associative.

It has recently been observed (Hyland and Schalk, 2000) that there is a faithful functor from the category Gam of simple games to the category Rel of sets and relations. In other words, every morphism

$$f : G \to G'$$

is represented by an ordinary (set theoretic) relation R_f between the positions of G and the positions of G'. If we also have $g : G' \to G''$ then we can find R_{gf} simply by composing the relation R_f with the relation R_g. This gives us a representation of strategies that is easier to work with than the traditional representation. For example, we already know that composition of relations is associative. We should like to be in the position that we can just define a strategy to be a relation of a certain kind; but we can't do that unless we know which relations are of the form R_f. The aim of this chapter is to find a necessary and sufficient condition for a relation (between the positions

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[1] This composition of strategies is due in essence – apparently independently – to Blass (1972) and Conway (1976). Joyal (1977) was the first to realise that Conway's games form a category in which morphisms are composed using this operation.

of one game and the positions of another) to represent a strategy. If we are lucky (and in this case we are), we shall be able to show directly that this condition is preserved by composition.

This chapter necessarily involves a great many rather tedious low-level manipulations; but it's worth it in the end, because we show how to define the category in a way that removes the need for such things.

Below we assume various well-known things about words and formal languages. A word over the alphabet Σ is an element of the free monoid Σ^* generated by Σ. The empty word, i.e. the unit of the free monoid, is denoted ε. Concatenation of words (which is composition in the free monoid) is represented by juxtaposition or sometimes a dot. From time to time we define languages using regular expressions, in the usual way. Finally we consider words to be ordered by the prefix relation – so $v \wedge w$ is the longest common prefix of v and w.

2.2 Simple Games

A (simple) game G consists of:

- a set G_O of Opponent moves and a disjoint set G_P of Proponent moves;
- a non-empty prefix-closed set of positions

$$\mathrm{pos}(G) \subseteq (G_O G_P)^*(\varepsilon + G_O).$$

We can divide pos(G) into two components

$$\begin{aligned} \mathrm{pos}_P(G) &= \mathrm{pos}(G) \cap (G_O G_P)^*, \\ \mathrm{pos}_O(G) &= \mathrm{pos}(G) \cap (G_O G_P)^* G_O. \end{aligned}$$

which we call *Proponent positions* (*P-positions*) and *Opponent positions* (*O-positions*) respectively.

Given a word $w \in (G_O \times G_P)^*$, where

$$w = (m_0, n_0)(m_1, n_1) \ldots (m_k, n_k),$$

we define $\hat{w} \in (G_O G_P)^*$ to be

$$\hat{w} = m_0 n_0 m_1 n_1 \dots m_k n_k.$$

A *pre-strategy* σ for G (written $\sigma : G$) is a prefix-closed subset of $(G_O \times G_P)^*$ such that for every w in σ, $\hat{w} \in \mathrm{pos}(G)$. The pre-strategy σ is a *strategy* if it is non-empty and *deterministic*: for every $v, w \in \sigma$,

$$\hat{v} \wedge \hat{w} \in (G_O G_P)^*.$$

Let $w \in (G_O \times G_P)^*$. Say $\hat{w} \in \mathrm{pos}_P(G)$ is *σ-reachable* iff $w \in \sigma$; and $\hat{w}.m \in \mathrm{pos}_O(G)$ is σ-reachable iff $w \in \sigma$.

2.2.1 Constructions on Games

We need to look at *restrictions* of plays: suppose w is a word over some alphabet which overlaps with $G_O \cup G_P$. Then $w|_G$ is defined to be the largest (scattered) subword of w consisting entirely of moves from G. Now we can define

$$\begin{aligned}
(G \otimes H)_O &= G_O + H_O \\
(G \otimes H)_P &= G_P + H_P \\
w \in \mathrm{pos}(G \otimes H) &\iff w|_G \in \mathrm{pos}(G) \text{ and} \\
&\quad\; w|_H \in \mathrm{pos}(H)
\end{aligned}$$

$$\begin{aligned}
(G \multimap H)_O &= G_P + H_O \\
(G \multimap H)_P &= G_O + H_P \\
w \in \mathrm{pos}(G \multimap H) &\iff w|_G \in \mathrm{pos}(G) \text{ and} \\
&\quad\; w|_H \in \mathrm{pos}(H)
\end{aligned}$$

This is a convenient place to prove a useful lemma, which perhaps also illustrates in minature how tedious it can be to work with these definitions.

Lemma 2.1. *Suppose* $w = v.m$ *and* $w' = v'.m'$ *are P-positions of* $G \multimap H$, *where* v, v' *are positions and* m, m' *are moves. If* $w|_G = w'|_G$ *and* $w|_H = w'|_H$ *then* $m = m'$.

Proof. We know that $|w|_G| + |w|_H| = |w|$ which is an even number, since w is a P-position. Therefore either $w|_G$ and $w|_H$ both have even length or they both have odd length. In the former case m is a Proponent move from either G or H, and is also a Proponent move from

$G \multimap H$. A Proponent move from $G \multimap H$ is (by definition) either an Opponent move from G or a Proponent move from H, hence it follows that m is a Proponent move from G. By a similar argument m' is also a Proponent move from G. Since $w|_G = w'|_G$ by hypothesis, we conclude that $m = m'$.

In the latter case (where $w|_G$ and $w|_H$ both have odd length), m and m' are both Opponent moves from G or H as well as being Opponent moves from $G \multimap H$. So they must in fact both be Opponent moves from H, and (since $w|_H = w'|_H$ by hypothesis) we again have $m = m'$. □

2.3 Pre-strategies as relations

Our first step is to show how *some* pre-strategies may be represented by relations, and to characterise the relations that represent them. Subsequently we shall establish that every strategy (but not every pre-strategy) is representable in this way. (The organisation of this section was suggested to the author by Peter Aczel.)

Let G and H be games, and let $K = G \multimap H$. We shall show how certain pre-strategies $\gamma : K$ can be represented by relations between $\mathrm{pos}(G)$ and $\mathrm{pos}(H)$. Recall that γ is a prefix-closed subset of the language $(K_O \times K_P)^*$; we define its relational counterpart

$$\mathrm{rel}(\gamma) = \{(\hat{w}|_G, \hat{w}|_H) \mid w \in \gamma\}. \tag{2.1}$$

We should be clear that we have made a particular choice of representation here, and have not excluded the possibility that there may be other reasonable representations. However this one turns out to work very well, and we aren't aware of any good alternatives to it. Of course we do not know *a priori* that the function rel is injective; in fact it is not, and one of our tasks is to find a subclass of pre-strategies on which rel *is* injective. (Of course we hope that this subclass will include all the actual strategies.)

2.3.1 Abstract Interlude

Here we shall abstract away a little from the situation at hand. Suppose only that we have a poset $(Q, \leq)$ in which every bounded interval[2] is finite, and a function $f : Q \to S$ for some set S. So as not to create unnecessary suspense, we reveal that our intended interpretations are:

$$\begin{aligned} Q &= \{w \in (K_O \times K_P)^* \mid \hat{w} \in \mathrm{pos}(K)\}, \\ S &= \mathrm{pos}(G) \times \mathrm{pos}(H), \\ f(w) &= (\hat{w}|_G, \hat{w}|_H). \end{aligned}$$

Let $\mathcal{L}(Q)$ be the set of lower sections (downward closed subsets) of Q, and let $\mathcal{P}(S)$ be the set of subsets of S. Both these are lattices ordered by inclusion. In our intended interpretation $\mathcal{L}(Q)$ is the set of pre-strategies and $\mathcal{P}(S)$ the set of relations.

We may define monotone functions

$$\begin{tikzcd} \mathcal{L}(Q) \arrow[r, shift left, "\mathrm{rel}"] & \mathcal{P}(S) \arrow[l, shift left, "\mathrm{str}"] \end{tikzcd}$$

as follows:

$$\begin{aligned} \mathrm{rel}(\gamma) &= \{f(w) \mid w \in \gamma\} \\ \mathrm{str}(R) &= \{w \in Q \mid (\forall v \leq w) f(v) \in R\} \end{aligned}$$

The choice of the symbols γ and R here emphasises the intended interpretation once more; we think of γ as being a pre-strategy, and R as a relation. The function rel here corresponds in the intended interpretation to (2.1) above. Also note that (as with rel) nothing has forced us to define str in this particular way. Intuitively we want str to be the inverse of rel 'enough of the time'.

The following proposition (noticed by Peter Aczel) gives us a convenient way to establish which pre-strategies are uniquely representable by relations, and which relations represent them.

Proposition 2.2. *There is an adjunction* $\mathrm{rel} \dashv \mathrm{str}$ *between the posets* $\mathcal{L}(Q)$ *and* $\mathcal{P}(S)$*. In other words*

$$\mathrm{rel}(\gamma) \subseteq R \iff \gamma \subseteq \mathrm{str}(R)$$

for all $\gamma \in \mathcal{L}(Q)$*,* $R \in \mathcal{P}(S)$*.*

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[2] By 'bounded interval' here I mean a set of the form $[x, z] = \{y \in Q \mid x \leq y \leq z\}$ for some pair x, z of elements of Q.

Proof. Since rel and str are monotone, it is sufficient to show that

$$\gamma \subseteq \mathrm{str}(\mathrm{rel}(\gamma)) \tag{2.2}$$

for every $\gamma \in \mathcal{L}(Q)$, and

$$\mathrm{rel}(\mathrm{str}(R)) \subseteq R \tag{2.3}$$

for every $R \in \mathcal{P}(S)$.

For (2.2), suppose $w \in \gamma$. Since γ is a lower section we know that $\{w' \mid w' \leq w\} \subseteq \gamma$, hence

$$\{f(w') \mid w' \leq w\} \subseteq \mathrm{rel}(\gamma)$$

and so $w \in \mathrm{str}(\mathrm{rel}(\gamma))$ as required.

For (2.3), suppose $x \in \mathrm{rel}(\mathrm{str}(R))$. Thus x is $f(w)$ for some w in $\mathrm{str}(R)$. But since $w \in \mathrm{str}(R)$ we know that $f(w) \in R$, so $x \in R$ as required. $\square$

It follows on simple general grounds that str and rel establish a bijection between the set

$$\{\gamma \in \mathcal{L}(Q) \mid \gamma = \mathrm{str}(\mathrm{rel}(\gamma))\}$$

of *stable pre-strategies* and the set

$$\{R \in \mathcal{P}(S) \mid R = \mathrm{rel}(\mathrm{str}(R))\}$$

of *stable relations*. (Intuitively a pre-strategy is stable if there is a unique relation representing it, and a relation is stable if it represents some (unique) representable pre-strategy.)

We can characterise the stable pre-strategies as follows.

Proposition 2.3. *Let $\gamma \in \mathcal{L}(Q)$. Then γ is stable just when it satisfies the following closure condition. If $v' \in Q$ and $f(v') = f(w)$ for some $w \in \gamma$ then*

$$(\forall v < v')[v \in \gamma] \;\Rightarrow\; v' \in \gamma.$$

Proof. First suppose that γ satisfies the closure condition. We must show that

$$v \in \mathrm{str}(\mathrm{rel}(\gamma)) \;\Rightarrow\; v \in \gamma \tag{2.4}$$

for each $v \in Q$. (We know from Prop. 2.2 that the converse is always true.)

We use induction on v. Take some $v' \in \mathrm{str}(\mathrm{rel}(\gamma))$ and suppose that (2.4) holds for all $v < v'$. Since $\mathrm{str}(\mathrm{rel}(\gamma))$ is downward closed, we know that each $v \in \mathrm{str}(\mathrm{rel}(\gamma))$, so $v \in \gamma$ by the inductive hypothesis. Also $v' \in \mathrm{str}(\mathrm{rel}(\gamma))$ implies that there is some $w \in \gamma$ with $f(w) = f(v')$. Thus the closure condition gives $w \in \gamma$ as required.

Next we show that the closure condition is a consequence of (2.4). In fact we show the contrapositive: suppose we have $v' \in Q$, $w \in \gamma$ with $f(v') = f(w)$ and $(\forall v < v')[v \in \gamma]$, but $v' \notin \gamma$. Now every $v \leq v'$ has $f(v) = f(w')$ for some $w' \in \gamma$, so $v' \in \mathrm{str}(\mathrm{rel}(\gamma))$. Hence v' fails to satisfy (2.4), which therefore does not hold in general. $\square$

We can also characterise the stable relations:

Proposition 2.4. $R \in \mathcal{P}(S)$ *is stable iff every* $x \in R$ *has a witness of stability* $w \in Q$, *with* $f(w) = x$ *and*

$$(\forall w' < w) f(w') \in R.$$

(This is just a restatement of the definition, so doesn't really need proving.)

2.3.2 Games Again

Now we concentrate on the intended interpretation. In this concrete setting we can find a simple characterisation of the stable relations. First, observe that the range of the function f (defined above) is the set

$$D \subseteq \big(\mathrm{pos}_O(G) \times \mathrm{pos}_O(H)\big) + \big(\mathrm{pos}_P(G) \times \mathrm{pos}_P(H)\big)$$

such that $(a, \varepsilon) \in D \Rightarrow a = \varepsilon$. Any stable relation must be a subset of D, of course. For any relation R we write

$$R_O = R \cap \big(\mathrm{pos}_O(G) \times \mathrm{pos}_O(H)\big),$$
$$R_P = R \cap \big(\mathrm{pos}_P(G) \times \mathrm{pos}_P(H)\big).$$

The next lemma characterises stable relations purely in terms of the order structure of the poset of positions. To state it, we need a simple definition.

Definition 2.5. In any poset, $v \lessdot v'$ means[3] that $v < v'$ and there is no x such that $v < x < v'$. We say 'v is covered by v''; the relation $\lessdot$ is the *covering relation*.

Remark: A poset is completely determined by its covering relation iff every bounded interval is finite. Any prefix-ordered set of finite words certainly has this property.

Lemma 2.6. *The relation* $R \subseteq D$ *is stable iff:*

- *for every* $(a, b) \in R_P$ *with* $b' \lessdot b$*,*

$$\begin{aligned} \textit{either} \quad & (\exists a' \lessdot a)\,(a', b') \in R \\ \textit{or} \quad & (\exists b'' \lessdot b')\,(a, b'') \in R. \end{aligned} \tag{2.5}$$

- *for every* $(a, b) \in R_O$ *with* $a' \lessdot a$*,*

$$\begin{aligned} \textit{either} \quad & (\exists b' \lessdot b)\,(a', b') \in R \\ \textit{or} \quad & (\exists a'' \lessdot a')\,(a'', b) \in R. \end{aligned} \tag{2.6}$$

Proof. Suppose first that R is stable, so it satisfies the condition in Prop. 2.3. Take $(a, b) \in R$, and let w be its witness of stability.

Suppose $(a, b) \in R_P$ with $b' \lessdot b$; so $b = b'.m$ for some move m in H_P. Since $f(\varepsilon) = (\varepsilon, \varepsilon)$ we know that $w \neq \varepsilon$, thus $w = w'.(n, n')$ for some $n \in K_O$, $n' \in K_P$. By the definition of f we know n' is the last move in either a or b; but it can't be the last move in a, which is from $G_P \subseteq K_O$, so in fact $n' = m$.

Also $f(w') \in R$ (because w is a witness of stability). We know n is in K_O, so either $n \in G_P$ or $n \in H_O$. In the former case $a = a'.n$ and $f(w') = (a', b')$; in the latter case $b' = b''.n$ and $f(w') = (a, b'')$. This establishes condition (2.5). The argument for condition (2.6) is similar.

Now suppose that R satisfies (2.5) and (2.6). We shall show that R is stable by finding a witness of stability for each $(a, b) \in R$ using induction on $\min\{|a|, |b|\}$.

Base case: if $a = \varepsilon$ then $b \in Q$ is the required witness. (If $b = \varepsilon$ then $a = \varepsilon$ too, since $R \subseteq D$.)

Inductive step: Suppose $a = a'.m$, $b = b'.n$ and that whenever $\min\{|p|, |q|\} < \min\{|a|, |b|\}$ (for $(p, q) \in R$) there is a corresponding

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[3] This notation is not standard. I have seen $-\!\!<$ and $\prec$ used for this, and doubtless other symbols have been used as well.

witness $w(p, q)$. So there are four possible situations, each with its corresponding witness for (a, b):

situation	witness
$(a', b') \in R_O$	$w(a', b').(m, n)$
$b' = b''.n',\ (a, b'') \in R_P$	$w(a, b'').(n', n)$
$(a', b') \in R_P$	$w(a', b').(n, m)$
$a' = a''.m',\ (a'', b) \in R_O$	$w(a'', b).(m', m)$

□

Corollary 2.7. *The identity relation is stable.*

Proof. Clearly every $(a, a) \in \mathrm{id}_G$ must satisfy the first disjunct in each of (2.5) and (2.6). □

Lemma 2.8. *Suppose* R *is stable and* $(a, b) \in R$.

- *If* $a' < a$ *with* $a' \in \mathrm{pos}_O(G)$ *then there is some* $(x, y) \in R$ *with* $a' \leq x \leq a$ *and* $y \leq b$*;*
- *If* $b' < b$ *with* $b' \in \mathrm{pos}_P(H)$ *then there is some* $(x, y) \in R$ *with* $x \leq a$ *and* $b' \leq y \leq b$*;*

with $(x, y) \neq (a, b)$ *in both cases. Moreover if we have both* a' *and* b' *then the two conditions can be satisfied simultaneously by the same* (x, y).

Proof. This follows from conditions (2.5) and (2.6). (Consider the cases $(a, b) \in R_P$ and $(a, b) \in R_O$ separately.) □

Definition 2.9. Suppose $(a, b), (c, d) \in R$. Say that (a, b) *crosses* (c, d) if either $a < c$ and $d < b$ or alternatively $c < a$ and $b < d$.

Now we can establish a really useful order-theoretic characterisation of stability.

Definition 2.10. The relation R is *downward complete* iff for every $(a, b) \in R$,

(i) if $a' \leq a$ for some $a' \in \mathrm{pos}_O(G)$ then there is $b' \in \mathrm{pos}_O(H)$ with $b' \leq b$ and $(a', b') \in R$,

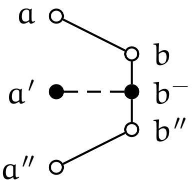
(ii) if $b'' \leq b$ for some $b'' \in \mathrm{pos}_P(H)$ then there is $a'' \in \mathrm{pos}_P(G)$ with $a'' \leq a$ and $(a'', b'') \in R$,

R is *cleanly* downward complete if

(iii) whenever we have both a' and b'' as above, it is possible to choose b' and a'' so that (a', b') does not cross (a'', b'').

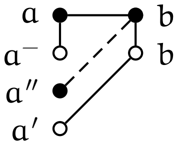
Proposition 2.11. *If the relation* R *is cleanly downward complete then it is stable.*

Proof. Suppose that R is cleanly downward complete: we shall show that it satisfies (2.5) and (2.6). Take $(a, b) \in R_P$ with $b^- \lessdot b$. There must be some $b'' \lessdot b^-$, hence $a'' \leq a$ with $(a'', b'') \in R$.



If $a'' = a$ then the second disjunct of (2.5) is satisfied; if not, there must be $a'' < a' < a$ with $a' \in \mathrm{pos}_O(G)$, so by condition (i) there is also $b' \in \mathrm{pos}_P(H)$ with $(a', b') \in R$. Furthermore we may assume that (a', b') does not cross (a'', b''), which implies that $b'' < b' < b$, thus $b' = b^-$. This satisfies the first disjunct of (2.5).

Now take $(a, b) \in R_O$ with $a^- \lessdot a$. There must be some $b' \lessdot b$, hence some $a' \leq a$ with $(a', b') \in R$.



If $a' = a^-$ then the first disjunct of (2.6) is satisfied; otherwise there must be some $a'' \in \mathrm{pos}_O(G)$ with $a' < a'' < a$. Then there is b'' in $\mathrm{pos}_O(H)$ with $(a'', b'') \in R$ such that (a', b') does not cross (a'', b''). It follows that $b' \leq b'' \leq b$, so $b'' = b$ and the second disjunct of (2.6) is satisfied. □

Proposition 2.12. *If* R *is stable then it is cleanly downward complete.*

Proof. Suppose that R satisfies (2.5) and (2.6). We shall show by induction on (a, b) that R is cleanly downward complete. Take some

$(a, b) \in R$, and suppose that conditions (i)–(iii) hold for all $(x, y) < (a, b)$ (i.e. $x \leq a$, $y \leq b$ and $(x, y) \neq (a, b)$). We'll show that they hold for (a, b) as well:

- i. Suppose $a' \leq a$ for some $a' \in \mathrm{pos}_O(G)$. If $a' = a$ then we take $b' = b$; otherwise $a' < a$ and by Lemma 2.8 there is $(x, y) \in R$ with $(x, y) < (a, b)$ and $a' \leq x$. Apply the inductive hypothesis.

- ii. Suppose $b'' \leq b$ for some $b'' \in \mathrm{pos}_P(H)$. If $b'' = b$ then we take $a'' = a$; otherwise $b'' < b$ and by Lemma 2.8 there is $(x, y) \in R$ with $(x, y) < (a, b)$ and $b'' \leq y$. Apply the inductive hypothesis.

- iii. Suppose we have both $a' \leq a$ and $b'' \leq b$. If $a' = a$ then a and b are O-positions, whence $a'' < a$ and $b'' < b$; similarly if $b'' = b$ then a and b are P-positions and $a' < a$, $b' < b$. In the remaining case we can use Lemma 2.8 to find $(x, y) \in R$ such that $(a', b'') \leq (x, y) < (a, b)$, and apply the inductive hypothesis.

□

Corollary 2.13. *The composite of two stable relations is stable.*

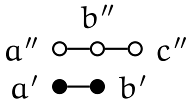
Proof. Suppose we have stable relations

$$\begin{aligned} R &\subseteq \mathrm{pos}(G) \times \mathrm{pos}(H), \\ S &\subseteq \mathrm{pos}(H) \times \mathrm{pos}(J), \end{aligned}$$

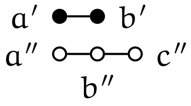
between games G, H and J. If $(a, c) \in RS$ then there is $b \in \mathrm{pos}(H)$ such that $(a, b) \in R$ and $(b, c) \in S$.

It is easy to establish conditions (i) and (ii): if $a' \leq a$ for some $a' \in \mathrm{pos}_O(G)$ then there is $b' \leq b$ such that $(a', b') \in R$, hence there is $c' \leq c$ such that $(b', c') \in R$. Thus $(a', c') \in R$ as required. If $c'' \leq c$ for some $c'' \in \mathrm{pos}_P(J)$ then there is $b'' \leq b$ with $(b'', c'') \in S$ and $a'' \leq a$ with $(a'', b'') \in R$.

If we have both a' and c'' as above, then let b'' be the maximum position in $\mathrm{pos}_P(H)$ such that $(b'', c'') \in S$. Now take $a'' \in \mathrm{pos}_P(G)$ and $b' \in \mathrm{pos}_O(H)$ such that (a', b') does not cross (a'', b''). There are two possibilities: if $a' < a''$ and $b' < b''$



then there must be some $c' < c$ such that $(b', c') \in S$, and we are done. If $a'' < a'$ and $b'' < b'$



then we know we can find $b''' \in \mathrm{pos}_P(H)$ and $c' \in \mathrm{pos}_P(J)$ such that $(b''', c'') \in S_P$ and $(b', c') \in S_O$, where (b''', c'') does not cross (b', c'). But b'' was chosen to be maximal, so we know that $b''' \leq b''$. Thus $c'' < c'$ and again we are done. $\square$

Therefore there is a category whose objects are games and whose morphisms are stable relations. This category does not seem to have very much structure; but we shall see that Gam – which certainly does have good structure – is a subcategory of it.

2.4 Strategies as Relations

The next problem is to determine which of the stable relations represent actual strategies, as opposed to mere pre-strategies. Before we do that, we need to make sure that every strategy is stable; for which we need a lemma.

Lemma 2.14. *Suppose* γ *is a strategy on* $G \multimap H$. *If* v *and* w *in* $\mathrm{pos}(G \multimap H)$ *are both* γ*-reachable and* $(v|_G, v|_H) = (w|_G, w|_H)$ *then* $v = w$.

Proof. Let v and w be γ-reachable and suppose that $v|_G = w|_G$ and $v|_H = w|_H$. Let x be the longest common prefix of v and w: they are obviously prefix incomparable, unless they're equal, since they have the same length. So if $v \neq w$ then $x.m \leq v$ and $x.n \leq w$ for some $m \neq n$. The only way this can happen is if $m \in G_O$ and $n \in H_P$ (or the other way round), so x is an O-position. But v and w are both γ-reachable, hence $x.m, x.n \in \gamma$. Since the longest common prefix of $x.m$ and $x.n$ must be a P-position we can only conclude that $m = n$, which contradicts the maximality of x. $\square$

Proposition 2.15. *Every strategy is stable.*

Proof. Let $\gamma : K$ be a strategy. We are going to use Prop. 2.3, so let v' be such that $\widehat{v'} \in \mathrm{pos}(K)$ and suppose that $f(v') = f(w)$ for some $w \in \gamma$. Suppose also that $v \in \gamma$ for every $v < v'$. In particular, $v' = v.(m, n)$ for some $v \in \gamma$, $m \in K_O$ and $n \in K_P$. By Lemma 2.1 we know that $w = w'.(m', n)$.

Now, the O-positions $\hat{v}.m$ and $\widehat{w'}.m'$ are both γ-reachable, so using Lemma 2.14 we may conclude that $\hat{v}.m = \widehat{w'}.m'$. It follows that $\widehat{v'} = \hat{w}$, hence $v' = w$. Thus $v' \in \gamma$ as required. □

The following defines the condition under which a stable relation represents a strategy.

Definition 2.16. The relation R is *coherent* iff for every (x, y) and (x', y') in R,

$$x \wedge x' \in \mathrm{pos}_P(G) \;\Rightarrow\; y \wedge y' \in \mathrm{pos}_P(H),$$

or equivalently

$$y \wedge y' \in \mathrm{pos}_O(H) \;\Rightarrow\; x \wedge x' \in \mathrm{pos}_O(G).$$

Proposition 2.17. *The pre-strategy γ is deterministic iff* $\mathrm{rel}(\gamma)$ *is coherent.*

Proof. γ is non-deterministic iff $\hat{v} \wedge \hat{w} \in \mathrm{pos}_O(K)$ for some $v, w \in \gamma$. This happens just when $\hat{v}|_G \wedge \hat{w}|_G$ is in $\mathrm{pos}_P(G)$ and $\hat{v}|_H \wedge \hat{w}|_H$ is in $\mathrm{pos}_O(H)$, which says that $\mathrm{rel}(\gamma)$ is incoherent. □

We should expect coherence to be preserved by composition, so the following is reassuring:

Proposition 2.18. *The composite of two coherent relations is coherent.*

Proof. Suppose we have coherent relations

$$\begin{aligned} R &\subseteq \mathrm{pos}(G) \times \mathrm{pos}(H), \\ S &\subseteq \mathrm{pos}(H) \times \mathrm{pos}(J), \end{aligned}$$

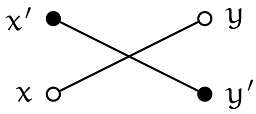
between games G, H and J. If (a, c), $(a', c') \in RS$ then there are b, $b' \in \mathrm{pos}(H)$ such that (a, b), $(a', b') \in R$ and (b, c), $(b', c') \in S$. If

$a \wedge a'$ is in $\mathrm{pos}_P(G)$ then $b \wedge b' \in \mathrm{pos}_P(H)$ because R is coherent, hence $c \wedge c' \in \mathrm{pos}_P(J)$ because S is coherent. So RS is coherent. □

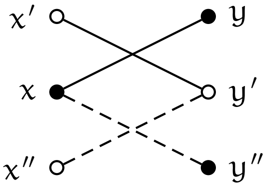
Curiously, condition (iii) of Def. 2.10 is redundant when R is coherent. In fact we have rather a strong property:

Proposition 2.19. *Suppose the relation* R *is downward complete and coherent. Then there are no crossings in* R*, so in particular* R *is cleanly downward complete.*

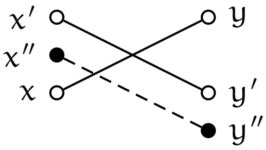
Proof. Suppose we have a counterexample, consisting of (x, y) and $(x', y') \in R$ such that (x, y) crosses (x', y'). We may suppose wlog that $x < x'$ and $y' < y$. Consider four cases in turn: if $(x, y) \in R_P$ and $(x', y') \in R_O$



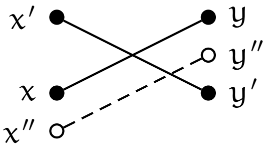
then $x \wedge x' \in \mathrm{pos}_P(G)$ but $y \wedge y' \in \mathrm{pos}_O(H)$, which contradicts coherence. If $(x, y) \in R_O$ and (x', y') is in R_P



then by Def. 2.10(i) there is some $y'' \leq y'$ such that $(x, y'') \in R$, and by condition (ii) there is some $x'' \leq x$ such that $(x'', y') \in R$. But now (x'', y') crosses (x, y''), which contradicts the previous case. The next possibility is that (x, y) and (x', y') are both in R_P



Since $x < x'$ there must be some $x'' \in \mathrm{pos}_O(G)$ between x and x', then by Def. 2.10(i) there is some $y'' \leq y'$ with $(x'', y'') \in R$. Thus (x, y) crosses (x'', y''), which is an instance of the first case. The final case is similar, as indicated by the following picture:



□

Putting it all together, a relation R represents a strategy iff it is downward complete and coherent.

2.4.1 Functoriality

We have shown that for any two games G and H, the set of strategies on $G \multimap H$ is in one-one correspondence with the set of coherent, downward complete relations included in the set D (defined at the beginning of §2.3.2). We have also shown that this condition is preserved by composition, so that our games form a category Gam.

If we are content to define composition of strategies in this way, then that's all we need. However we should also check (just this once) that our composition is the same as the traditional version. The traditional definition of composition is usually referred to as 'parallel composition + hiding', by analogy with CCS (Milner, 1980). The neatest way to describe it uses a simple auxiliary notion:

Definition 2.20. Suppose we have $\sigma : G \multimap H$ and $\tau : H \multimap J$. A *combined sequence* for σ and τ is a word w over the combined move alphabets of G, H and J such that $w|_{G+H} \in \sigma$ and $w|_{H+J} \in \tau$. Although we don't use the fact, it is enlightening to observe that any such word w must be a prefix of some word of the form

$$(J_O(H_O(G_OG_P)^*H_P)^*J_P)^*.$$

The composite $\tau\sigma$ is then defined to be the set

$$\{w|_{G+J} \mid w \text{ is a combined sequence for } \sigma \text{ and } \tau\}.$$

It isn't obvious that this even defines a strategy, but that follows from the connection with our relational representation, so thankfully we don't have to prove it directly. We need to show that

$$\operatorname{rel}(\tau\sigma) = \operatorname{rel}(\sigma);\operatorname{rel}(\tau).$$

One direction is almost trivial:

Proposition 2.21. *If* $(x, z) \in \operatorname{rel}(\tau\sigma)$ *then there exists some* y *in* $\operatorname{pos}(H)$ *such that* $(x, y) \in \operatorname{rel}(\sigma)$ *and* $(y, z) \in \operatorname{rel}(\tau)$.

Proof. Suppose $(x, z) \in \operatorname{rel}(\tau\sigma)$. Then there is a word $v \in \tau\sigma$ such that $\hat{v}|_G = x$ and $\hat{v}|_J = z$. By the definition of composition above, there must be some combined sequence w for σ and τ such that $w|_{G+J} = \hat{v}$. So we have $(w|_G, w|_H) \in \sigma$ and $(w|_H, w|_J) \in \tau$ as required. $\square$

The other direction is a little harder; we strengthen the hypothesis and use induction.

Definition 2.22. An *extended combined sequence* for σ and τ is a word w over the combined move alphabets of G, H and J such that $w|_{G+H}$ is σ-reachable and $w|_{H+J}$ is τ-reachable.

Definition 2.23. Given a strategy $\sigma : G \multimap H$, define the relation $\mathrm{erel}(\sigma) \subseteq \mathrm{pos}(G) \times \mathrm{pos}(H)$ to be

$$\{(w|_G, w|_H) \mid w \text{ is } \sigma\text{-reachable}\}.$$

(Note that $\mathrm{rel}(\sigma) = \mathrm{erel}(\sigma) \cap D$, where D is the set defined at the beginning of §2.3.2.)

The following easy lemma is needed.

Lemma 2.24. *If* $(x, y) \in \mathrm{erel}(\sigma)$ *with* $x \in \mathrm{pos}_O(G)$ *then* $y \in \mathrm{pos}_O(H)$.

Proof. Suppose (for a contradiction) that $x \in \mathrm{pos}_O(G)$ and y is in $\mathrm{pos}_P(H)$. Let w be the σ-reachable word with $w|_G = x$ and $w|_H = y$, then $|w| = |x| + |y|$, which is odd+even hence odd. So the last move in w must be from $(G \multimap H)_O$, i.e. either from G_P or H_O. But x is an O-position, so the last move in x is from G_O; and y is a P-position, so the last move in y is from G_P. This is a contradiction. $\square$

Proposition 2.25. *Suppose* $(x, y) \in \mathrm{erel}(\sigma)$ *and* $(y, z) \in \mathrm{erel}(\tau)$. *Then there is an extended combined sequence* w *for* σ *and* τ *such that* $w|_G = x$, $w|_H = y$ *and* $w|_J = z$.

Proof. We use induction on the triple (x, y, z). The base case is that $x = y = z = \varepsilon$, in which case we can just take the empty sequence. Now suppose that the result is true for all $(a, b, c) < (x, y, z)$.

By Lemma 2.24, there are four possibilities for x, y and z. The first is that $x \in G_P$, $y \in H_P$ and $z \in J_P$, which we abbreviate as ○○○ because the G, H, J projections are all P-positions. Using similar abbreviations, the other three cases are ○○●, ○●● and ●●●. We consider each case separately:

(○○○) If $z = \varepsilon$ then $x = y = \varepsilon$ too, so suppose that $z = z'.m$ for some move m. Clearly $(y, z') \in \mathrm{erel}(\tau)$, so let w' be the extended combined sequence for (x, y, z'). Let $w = w'.m$.

(○○●) The word representing (y, z) has its last move from J_O or H_P. In the former case, $z = z'.m$ for some move m and pair (y, z') in $\mathrm{erel}(\tau)$, so let w' be the extended combined sequence representing (x, y, z') and define $w = w'.m$. In the latter case, y is equal to $y'.m$ for some move m with $(x, y') \in \mathrm{erel}(\sigma)$ and $(y', z) \in \mathrm{erel}(\tau)$, so let w' be the extended combined sequence representing (x, y', z) and again define $w = w'.m$.

(○●●) The word representing (x, y) has its last move from H_O or G_P. In the former case, $y = y'.m$ for some move m with (x, y') in $\mathrm{erel}(\sigma)$ and $(y', z) \in \mathrm{erel}(\tau)$, so let w' be the extended combined sequence representing (x, y', z) and define $w = w'.m$. In the latter case, $x = x'.m$ for some move m with (x', y) in $\mathrm{erel}(\sigma)$ so let w' be the extended combined sequence representing (x', y, z) and again define $w = w'.m$.

(●●●) Here $x = x'.m$ for some move m, so let w' be the extended combined sequence representing (x', y, z) and set $w = w'.m$.

□

Corollary 2.26. *Suppose* $(x, y) \in \mathrm{rel}(\sigma)$ *and* $(y, z) \in \mathrm{rel}(\tau)$. *Then there is a combined sequence* w *for* σ *and* τ *such that* $w|_G = x$, $w|_H = y$ *and* $w|_J = z$.

So we have established that composing two strategies in the traditional way gives the same result as our method of composing the corresponding relations, hence there is a faithful functor from Gam to the category Rel of sets and relations.

2.5 Strategies as Hypergraphs

Let us recall the category of hypergraphs (Melliès, 2003).

Definition 2.27. A hypergraph X consists of:

- a set X^+ of *positive atoms*, and a disjoint set X^- of *negative atoms*; together with

- a set $\Gamma(X)$ of non-empty finite subsets of $X^+ \cup X^-$, containing all singletons, called the *atomic coherence*.

We also define the *strict atomic coherence* $\Gamma^*(X)$ to be the subset of $\Gamma(X)$ containing all its elements except the singletons.

A hypergraph morphism $R : X \to Y$ consists of a pair of relations

$$R^+ \subseteq X^+ \times Y^+$$

and

$$R^- \subseteq X^- \times Y^-$$

such that for every $S \subseteq (R^+ \cup R^-)$,

$$\pi_1[S] \in \Gamma(X) \;\Rightarrow\; \pi_2[S] \in \Gamma(Y) \tag{2.7}$$

and

$$\pi_1[S] \in \Gamma^*(X) \;\Rightarrow\; \pi_2[S] \in \Gamma^*(Y). \tag{2.8}$$

Notice that in the presence of (2.7), the right-hand side of (2.8) may equivalently be replaced by the condition $|\pi_2[S]| > 1$.

In this section we show how to give D the structure of a hypergraph in such a way that coherent relations coincide exactly with hypergraph morphisms. The first step is to show that Def. 2.16 is equivalent to a stronger-looking condition.

Proposition 2.28. R *is coherent iff for every subset* $S \subseteq R$*,*

$$\bigwedge \pi_1[S] \in \mathrm{pos}_p(G) \;\Rightarrow\; \bigwedge \pi_2[S] \in \mathrm{pos}_p(H). \tag{2.9}$$

Clearly (2.9) implies coherence; it's the converse that is not quite immediate. For that we shall use the following lemmas:

Lemma 2.29. *Suppose we have a set divided into four quadrants:*

$$\begin{array}{c|c} NW & NE \\ \hline SW & SE \end{array}$$

and we know that

- *the North is inhabited,*
- *the South is inhabited,*

- *the West is inhabited,*
- *the East is inhabited;*

Then either the NW and SE are both inhabited, or the NE and SW are both inhabited.

Proof. If all four quadrants are inhabited, of course the conclusion follows. Otherwise there is an uninhabited quadrant, say the NW. But the North is inhabited so the NE must be, and the West is inhabited so the SW must be. (A symmetrical argument applies to the other three quadrants.) ☐

Lemma 2.30. *Suppose we are given two sets* X *and* Y*, and an equivalence relation* $\sim$ *on each set. We also have* $S \subseteq X \times Y$*. The following are equivalent:*

1. *There are* (x, y)*,* $(x', y') \in S$ *such that* $x \nsim x'$ *and* $y \nsim y'$*.*
2. *There are* (x, y)*,* $(x', y') \in S$ *such that* $x \nsim x'$*; and there are also* (v, w)*,* $(v', w') \in S$ *such that* $w \nsim w'$*.*

Proof. Of course (1) implies (2) immediately. For the converse, suppose we have (x, y), (x', y'), (v, w) and (v', w') as in (2). Now partition S into four quadrants; the element (m, n) is placed according to the following:

$$\begin{array}{c|c} x \sim m & x \nsim m \\ w \sim n & w \sim n \\ \hline x \sim m & x \nsim m \\ w \nsim n & w \nsim n \end{array}$$

We know the North is inhabited by (v, w), the South by (v', w'), the West by (x, y) and the East by (x', y'). So by Lemma 2.29 there must be two diagonally-opposite pairs, say (a, b) and (a', b'). Since they are in different columns we know $a \nsim a'$, and since they are in different rows, $b \nsim b'$. So (1) is satisfied. ☐

That is the general fact we need for Prop. 2.28. We also need the following (very easy) fact about prefix orderings (which also applies more generally to tree-like posets).

Lemma 2.31. *Suppose* $x \in S \subseteq A^*$ *for some alphabet* A*. There is* $y \in S$ *such that* $\bigwedge S = x \wedge y$*.*

Proof. If $\bigwedge S \in S$ then take $y = \bigwedge S$. Otherwise there must be $y \in S$ such that $(\bigwedge S).a \leq x$ and $(\bigwedge S).b \leq y$ for $a \neq b$ in A. Thus $x \wedge y$ is equal to $\bigwedge S$. □

Proof of Prop. 2.28. Suppose we have a relation R and a subset S of R. Let $x = \bigwedge \pi_1[S]$ and $y = \bigwedge \pi_2[S]$. Assume that S violates (2.9), so x is a P-position but y is an O-position. If $(x, z) \in S$ for some z then by Lemma 2.31 there is $(x', y') \in S$ with $y = z \wedge y'$. But of course $x \wedge x' = x$, and this contradicts coherence. If $(w, y) \in S$ for some w then a similar argument applies.

The remaining possiblity is that $x \notin \pi_1[S]$ and $y \notin \pi_2[S]$. For every $(w, z) \in S$ we know that there is some move m such that $x.m \leq w$, and some move n such that $y.n \leq z$. This allows us to define equivalence relations $\sim$ on $\pi_1[S]$ and $\pi_2[S]$ by letting $w \sim w'$ iff $x.m \leq w \wedge w'$ for some move m, and $z \sim z'$ iff $y.n \leq z \wedge z'$ for some move n. Clearly there must be $w', w \in \pi_1[S]$ with $w \nsim w'$ (since there is no move m such that $x.m \leq \bigwedge \pi_1[S]$) and similarly there must be $z, z' \in \pi_2[S]$ with $z \nsim z'$. Then by Lemma 2.30 we know that there are $(w, z), (w', z') \in S$ with $w \wedge w' = x$ and $z \wedge z' = y$, which contradicts coherence. □

Thus we can give each set $P = \text{pos}(G)$ the structure of a hypergraph; we set $P^+ = \text{pos}_P(G)$ and $P^- = \text{pos}_O(G)$, then decree $\Gamma(P)$ to consist of all sets $S \subseteq P$ for which $\bigwedge S \in \text{pos}_P(G)$, together with all singletons. Prop. 2.28 shows that a coherent relation is one that satisfies condition (2.7). We now show that, as a consequence of the downward completeness restriction, condition (2.8) is automatically satisfied too.

Lemma 2.32. *Let* R *be a coherent stable relation, and suppose* $S \subseteq R$. *If* $\bigwedge \pi_1[S] \in \text{pos}_P(G)$ *and* $\pi_2[S]$ *is totally ordered by* $\leq$ *then* $\pi_1[S]$ *is also totally ordered.*

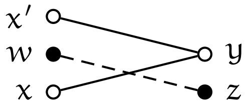
Similarly if $\bigwedge \pi_2[S] \in \text{pos}_O(H)$ *and* $\pi_1[S]$ *is totally ordered then* $\pi_2[S]$ *is totally ordered.*

Proof. We'll just prove the first part, since the second is similar. Suppose (for contradiction) that $\bigwedge \pi_1[S] \in \text{pos}_P(G)$ and $\pi_2[S]$ is totally ordered, and there are incomparable elements x and x' in $\pi_1[S]$.

If x, x' are incomparable then neither can be ε, so there must be $w \lessdot x$ and $w' \lessdot x'$. We know that $x \wedge x'$ is a P-position, so every pair from $\{w, x\} \times \{w', x'\}$ has the same meet. At least one of these pairs consists of two O-positions, so there are O-postions v and v' such that $v \wedge v' \in \mathrm{pos}_P(G)$. By Prop. 2.12 there are $z \leq y$ and $z' \leq y$ such that (v, z), $(v', z') \in R$. But z and z' must be comparable since both are prefixes of y, thus $z \wedge z'$ is an O-position. This contradicts coherence. □

Proposition 2.33. *Let* R *be a coherent stable relation, and suppose* $S \subseteq R$. *If* $\bigwedge \pi_1[S] \in \mathrm{pos}_P(G)$ *and* $|\pi_2[S]| = 1$ *then* $|\pi_1[S]| = 1$.

Proof. Again we'll just prove the first part. Suppose (for contradiction) that $\bigwedge \pi_1[S] \in \mathrm{pos}_P(G)$ and $\pi_2[S] = \{y\}$, and that there are $x, x' \in \pi_1[S]$ with $x \neq x'$ and $x \wedge x' = \bigwedge \pi_1[S]$. Lemma 2.32 tells us that x and x' are comparable, so suppose $x \leq x'$.



Thus there is $w \in \mathrm{pos}_O(G)$ with $x < w < x'$. By Prop. 2.12 there is $z \leq y$ such that $(x, z) \in R$. But this is a crossing, which contradicts Prop. 2.19. □

So, we have shown that for any game G, the set $\mathrm{pos}(G)$ can be structured as a hypergraph; and then for any two games G and H, the set of strategies on $G \multimap H$ is in one-one correspondence with the set of downward complete relations between $\mathrm{pos}(G)$ and $\mathrm{pos}(H)$ that also preserve the hypergraph structure. In other words, there is a faithful functor $\mathcal{H}$ from Gam to the category HGraph of hypergraphs. So there is some sense in which a simple game is a tree-structured, alternating hypergraph.

(There's also a faithful functor from HGraph to Ehrhard's category HCoh of hypercoherences (see Def. 3.12), which just forgets the polarisation of atoms; so it's trivially also true that games can be faithfully represented by hypercoherences.)

Proposition 2.34. *The functor* $\mathcal{H}$ *is monoidal (in the lax sense – see the next section for a definition).*

Proof sketch. Let I be the game with no moves, whose sole position is ε. Then $\mathcal{H}(I)$ has just one (positive) atom, so there's an isomorphism $m_I : I \to \mathcal{H}(I)$. Now let X and Y be games. We can define a hypergraph morphism

$$m_{X,Y} : \mathcal{H}(X) \otimes \mathcal{H}(Y) \to \mathcal{H}(X \otimes Y)$$

that consists of all pairs $((p|_X, p|_Y), p)$ for $p \in \text{pos}(X \otimes Y)$. This family of morphisms is natural in X and Y, and preserves the isomorphisms that give the categories Gam and HGraph their monoidal structure. $\square$

Chapter 3

Completion Principles

3.1 Models of linear logic

First we review the categorical structures that are needed to model linear logic, giving nearly all of the necessary details. The notion of model that we give here was discovered by a number of authors over a period of several years, notably Seely (1989) and Bierman (1995).

The fundamental structure that we need to model multiplicative intuitionistic linear logic (MILL) is that of a symmetric monoidal closed category. A symmetric monoidal category is a category $\mathbb{C}$ equipped with a distinguished object I and a functor

$$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C},$$

together with natural isomorphisms with components:

$$\begin{aligned} \alpha_{A,B,C} &: (A \otimes B) \otimes C \to A \otimes (B \otimes C) \\ \lambda_A &: A \to I \otimes A \\ \rho_A &: A \to A \otimes I \\ \sigma_{A,B} &: A \otimes B \to B \otimes A \end{aligned}$$

such that $\lambda_I = \rho_I$, $\sigma^{-1}_{A,B} = \sigma_{B,A}$, and the following commute:

$$\begin{tikzcd} A \arrow[r, "\rho_A"] \arrow[dr, "\lambda_A"'] & A \otimes I \arrow[d, "\sigma_{I,A}"] \\ & I \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \otimes B \arrow[r, "\rho_A \otimes B"] \arrow[dr, "A \otimes \lambda_B"'] & (A \otimes I) \otimes B \arrow[d, "\alpha_{A,I,B}"] \\ & A \otimes (I \otimes B) \end{tikzcd}$$

$$\begin{tikzcd}
& (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B\otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[dd, "A \otimes \alpha_{B,C,D}"] \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A,B,C\otimes D}"] \arrow[dr, "\alpha_{A\otimes B,C,D}"'] & & \\
& (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A,B,C\otimes D}"'] & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[r, "\sigma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha_{B,C,A}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "\sigma_{A,B} \otimes C"'] & & & B \otimes (C \otimes A) \\
& (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[ur, "B \otimes \sigma_{A,C}"'] &
\end{tikzcd}$$

In such diagrams it is common to write the identity morphism on an object X simply as X, as we've done above.

Note that if $\mathbb{C}$ and $\mathbb{D}$ are symmetric monoidal categories, then the product $\mathbb{C} \times \mathbb{D}$ has a natural symmetric monoidal structure defined pointwise, i.e. for $X, X' \in \mathbb{C}$ and $Y, Y' \in \mathbb{D}$, the tensor $(X, Y) \otimes (X', Y')$ is defined to be $(X \otimes X', Y \otimes Y')$.

A symmetric monoidal category has the structure that we need to interpret Girard's tensor connective. There are four reasonable senses (useful in different situations) in which a functor between two symmetric monoidal categories can considered as respecting the monoidal structure. Let $\mathbb{C}$ and $\mathbb{D}$ be symmetric monoidal categories. Then in non-decreasing order of strictness:

- A (*lax*) *monoidal functor* is a functor $F : \mathbb{C} \to \mathbb{D}$ together with an arrow $m^F_I : I \to F(I)$ in $\mathbb{D}$ and a natural transformation m^F from $\lambda X, Y.F(X) \otimes F(Y)$ to $\lambda X, Y.F(X \otimes Y)$. We omit the superscript F unless there is a risk of ambiguity. The following are required to commute:

$$\begin{tikzcd}
FX \arrow[r, "F\rho_X"] \arrow[d, "\rho_{FX}"'] & F(X \otimes I) \\
FX \otimes I \arrow[r, "FX \otimes m_I"'] & FX \otimes FI \arrow[u, "m_{X,I}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes FY \arrow[r, "m_{X,Y}"] \arrow[d, "\sigma_{FX,FY}"'] & F(X \otimes Y) \arrow[d, "F\sigma_{X,Y}"] \\
FY \otimes FX \arrow[r, "m_{Y,X}"'] & F(Y \otimes X)
\end{tikzcd}$$

$$\begin{tikzcd}
(FX \otimes FY) \otimes FZ \arrow[r, "m_{X,Y} \otimes FZ"] \arrow[d, swap, "\alpha_{FX,FY,FZ}"] & F(X \otimes Y) \otimes FZ \arrow[r, "m_{X \otimes Y,Z}"] & F((X \otimes Y) \otimes Z) \arrow[d, "F\alpha_{X,Y,Z}"] \\
FX \otimes (FY \otimes FZ) \arrow[r, swap, "FX \otimes m_{Y,Z}"] & FX \otimes F(Y \otimes Z) \arrow[r, swap, "m_{X,Y \otimes Z}"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

(The ‘missing’ diagram for λ is redundant in the symmetric case, since λ can be defined in terms of ρ and σ.)

- An *oplax monoidal functor* is a functor $F : \mathbb{C} \to \mathbb{D}$ such that $F^{op} : \mathbb{C}^{op} \to \mathbb{D}^{op}$ is a monoidal functor.

- A *strong monoidal functor* is a monoidal functor $F : \mathbb{C} \to \mathbb{D}$ such that the arrow m_I is an isomorphism and each component $m_{X,Y}$ is an isomorphism too.

- A *strict monoidal functor* is a functor F such that for any objects $X, Y \in \mathbb{C}$ we have $F(X \otimes Y) = FX \otimes FY$, and for any arrows $f, g : X \to Y$ we have $F(f \otimes g) = Ff \otimes Fg$. Strict monoidal functors don’t arise very often in practice, but when they do they’re particularly easy to deal with.

(We are departing slightly from standard terminology here, in that what we call a monoidal functor is often called a (lax) *symmetric* monoidal functor, because we require commutativity of the diagram involving σ. However we have no use for non-symmetric monoidal functors, so we drop the adjective.)

We’re mainly interested in the lax case in this report. Many natural functors are monoidal. The ones we need are:

- The identity functor $1 : \mathbb{C} \to \mathbb{C}$ is clearly strict monoidal;

- The constant functor with value I is strong monoidal. Here m_I is the identity on I and $m_X, Y : I \otimes I \to I$ is λ_I^{-1} (which is the same as ρ_I^{-1}). Note that constant functors are not generally monoidal.

- The tensor functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is strong monoidal. (Note that $\mathbb{C} \times \mathbb{C}$ is symmetric monoidal, the structure given pointwise.) Here m_I is λ_I (which is equal to ρ_I) and

$$m_{(X,X'),(Y,Y')} : (X \otimes X') \otimes (Y \otimes Y') \to (X \otimes Y) \otimes (X' \otimes Y')$$

is the appropriate combination of associativity (α) and symmetry (σ) isomorphisms.

- The composite GF of two monoidal functors F and G is again monoidal, with m_I^{GF} equal to

$$I \xrightarrow{m_I^G} GI \xrightarrow{Gm_I^F} GFI$$

and $m_{X,Y}^{GF}$ equal to

$$GFX \otimes GFY \xrightarrow{m^G_{FX,FY}} G(FX \otimes FY) \xrightarrow{Gm^F_{X,Y}} GF(X \otimes Y).$$

(Of course one needs to check that this does indeed give a legitimate monoidal functor, which is not entirely obvious.)

- Combining the two previous examples, if F and G are monoidal functors then so is $\lambda X.FX \otimes GX$. Explicitly m_I is equal to

$$I \xrightarrow{\lambda_I} I \otimes I \xrightarrow{m_I^F \otimes m_I^G} FI \otimes GI$$

and $m_{X,Y}$ is

$$\begin{tikzcd}
(FX \otimes GX) \otimes (FY \otimes GY) \arrow[d, "\theta"] \\
(FX \otimes FY) \otimes (GX \otimes GY) \arrow[d, "m^F_{X,Y} \otimes m^G_{X,Y}"] \\
F(X \otimes Y) \otimes G(X \otimes Y)
\end{tikzcd}$$

where θ is the (unique) appropriate combination of structural isomorphisms. We continue to use the letter θ as a short-hand for this particular natural isomorphism.

Given monoidal functors $F, G : \mathbb{C} \longrightarrow \mathbb{D}$, a monoidal natural transformation ϕ from F to G is a natural transformation such that

$$\begin{tikzcd}
 & FI \arrow[dd, "\phi_I"] \\
I \arrow[ur, "m_I^F"] \arrow[dr, "m_I^G"'] & \\
 & GI
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes FY \arrow[r, "m^F_{X,Y}"] \arrow[d, "\phi_X \otimes \phi_Y"'] & F(X \otimes Y) \arrow[d, "\phi_{X\otimes Y}"] \\
GX \otimes GY \arrow[r, "m^G_{X,Y}"'] & G(X \otimes Y)
\end{tikzcd}$$

commute.

For the category to be symmetric monoidal *closed* means further that for every object B, the functor $\lambda A.A \otimes B$ has a right adjoint which we denote $\lambda C.B \multimap C$. As with any parameterised adjunction, it follows (Mac Lane, 1978, Theorem IV.7.3) that we can regard $\multimap$ as a functor $\mathbb{C}^{\mathrm{op}} \otimes \mathbb{C} \to \mathbb{C}$ so that there's an isomorphism

$$\mathbb{C}(A \otimes B, C) \cong \mathbb{C}(A, B \multimap C) \tag{3.1}$$

natural in all three variables. This allows a natural interpretation of the linear implication. Passing rightwards across the isomorphism (3.1) is called 'currying' and the converse process is 'uncurrying'.

The units and counits of our adjunctions are given by natural transformations η^B and ε^B with components

$$\begin{aligned} \eta^B_A &: A \to B \multimap (A \otimes B) \\ \varepsilon^B_C &: (B \multimap C) \otimes B \to C \end{aligned}$$

The components ε^B_C of the counit are called 'evaluation maps', because they can be regarded as evaluating a function of type $B \multimap C$ at its argument of type B. (By analogy, one sometimes sees the components η^B_A called 'coevaluation maps'.)

To model classical multiplicative linear logic (MLL), we also need an interpretation of the linear negation 'perp'. The structure required here is that of a *$*$-autonomous category* (Barr, 1979). There are several equivalent definitions of this notion; one of the simplest is to ask for a symmetric monoidal closed category equipped with a distinguished object $\perp$ such that for every object B, the arrow

$$\zeta^B_\perp : B \to (B \multimap \perp) \multimap \perp$$

obtained by currying the composite

$$B \otimes (B \multimap \perp) \xrightarrow{\sigma_{B,B \multimap \perp}} (B \multimap \perp) \otimes B \xrightarrow{\varepsilon^{B}_{\perp}} \perp$$

is an isomorphism. We define $X^{\perp}$ to be $X \multimap \perp$, and (by definition) we have $X^{\perp\perp} \cong X$.

Thus a $*$-autonomous category suffices to model the multiplicative connective. The additive connectives are very easy: we need finite products to model & ('with') and finite coproducts to model $\oplus$ ('plus'). In the $*$-autonomous case either of these implies the existence of the other: for example if we have a $*$-autonomous category with finite products then it has coproducts too, given by the de Morgan equation

$$A + B = (A^{\perp} \times B^{\perp})^{\perp}.$$

The most complicated structure is the one needed to give a sound interpretation of Girard's 'of course' modality. There is actually a reasonably short definition of the needed structure, viz:

Definition 3.1 (Maneggia (2004)). A *linear exponential comonad* ! on a symmetric monoidal closed category $\mathbb{C}$ is a monoidal comonad such that the induced tensor product on the (Eilenberg-Moore) category of coalgebras for the comonad is in fact a categorical product.

However it is not at all obvious that this definition gives the structure that we intuitively need, and it is worth analysing it in detail. In particular it is useful to work out exactly what structure is needed on the category $\mathbb{C}$ itself in order to describe such a comonad. Breaking the definition down into basic parts, we need:

- A functor $! : \mathbb{C} \to \mathbb{C}$,
- structure to make the functor monoidal, specifically
  - an arrow $m_I : I \to\ !I$,
  - for every $X, Y \in \mathbb{C}$, an arrow

$$m_{X,Y} : !X \otimes !Y \to\ !(X \otimes Y)$$

  such that the arrows $m_{X,Y}$ and m_I commute with the structural isomorphisms of the monoidal category in the sense that the following three diagrams commute:

$$\begin{tikzcd}
!X \arrow[r, "!\rho_X"] \arrow[d, "\rho_{!X}"'] & !(X \otimes I) \\
!X \otimes I \arrow[r, "!X \otimes m_I"'] & !X \otimes !I \arrow[u, "m_{X,I}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
!X \otimes !Y \arrow[r, "m_{X,Y}"] \arrow[d, "\sigma_{!X,!Y}"'] & !(X \otimes Y) \arrow[d, "!\sigma_{X,Y}"] \\
!Y \otimes !X \arrow[r, "m_{Y,X}"] & !(Y \otimes X)
\end{tikzcd}$$

$$\begin{tikzcd}
(!W \otimes !X) \otimes !Y \arrow[r, "\alpha_{!W,!X,!Y}"'] \arrow[d, "m_{W,X} \otimes !Y"'] & !W \otimes (!X \otimes !Y) \arrow[d, "!W \otimes m_{X,Y}"] \\
!(W \otimes X) \otimes !Y \arrow[d, "m_{W \otimes X,Y}"'] & !W \otimes !(X \otimes Y) \arrow[d, "m_{W,X \otimes Y}"] \\
!((W \otimes X) \otimes Y) \arrow[r, "!\alpha_{W,X,Y}"'] & !(W \otimes (X \otimes Y))
\end{tikzcd}$$

- natural transformations[1] $\varepsilon_X : !X \to X$ and $\delta_X : !X \to !!X$. These natural transfromations must be monoidal, which means that the diagrams below must commute:

$$\begin{tikzcd}
!X \otimes !Y \arrow[r, "m_{X,Y}"] \arrow[d, "\varepsilon_X \otimes \varepsilon_Y"'] & !(X \otimes Y) \arrow[d, "\varepsilon_{X \otimes Y}"] \\
X \otimes Y \arrow[r, "1"'] & X \otimes Y
\end{tikzcd}$$

$$\begin{tikzcd}
!X \otimes !Y \arrow[rr, "m_{X,Y}"] \arrow[d, "\delta_X \otimes \delta_Y"'] & & !(X \otimes Y) \arrow[d, "\delta_{X \otimes Y}"] \\
!!X \otimes !!Y \arrow[r, "m_{!X,!Y}"'] & !(!X \otimes !Y) \arrow[r, "!m_{X,Y}"'] & !!(X \otimes Y)
\end{tikzcd}$$

---

[1] There is an apparent clash of notation here, in that we have already used (and will later use) ε_A^B for the evaluation maps. However the evaluation maps are distinguished by the presence of a superscript, and furthermore the context will always make it clear which we mean.

$$\begin{tikzcd}
I \arrow[r, "m_I"] \arrow[dr, "1"'] & !I \arrow[d, "\varepsilon_I"] \\
& I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "m_I"] \arrow[d, "m_I"'] & !I \arrow[d, "\delta_I"] \\
!I \arrow[r, "!m_I"'] & !!I
\end{tikzcd}$$

- Also $(!, \varepsilon, \delta)$ must be a comonad in the usual sense, which means that the three diagrams below must commute.

$$\begin{tikzcd}
!X \arrow[r, "\delta_X"] \arrow[dr, "1"'] & !!X \arrow[d, "\varepsilon_{!X}"] \\
& !X
\end{tikzcd}
\qquad
\begin{tikzcd}
!X \arrow[r, "\delta_X"] \arrow[dr, "1"'] & !!X \arrow[d, "!\varepsilon_X"] \\
& !X
\end{tikzcd}$$

$$\begin{tikzcd}
!X \arrow[r, "\delta_X"] \arrow[d, "\delta_X"'] & !!X \arrow[d, "!\delta_X"] \\
!!X \arrow[r, "\delta_{!X}"'] & !!!X
\end{tikzcd}$$

- The most subtle requirement concerns the induced tensor product on the category of coalgebras.

  First recall that a coalgebra for a comonad ! consists of an object $X \in \mathbb{C}$ together with a map $\alpha : X \to !X$ such that the composite

$$\begin{tikzcd}
X \arrow[r, "\alpha"] & !X \arrow[r, "\varepsilon_X"] & X
\end{tikzcd}$$

  is the identity on X, and the diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & !X \arrow[d, "!\alpha"] \\
!X \arrow[r, "\delta_X"'] & !X
\end{tikzcd}$$

  commutes. If we have coalgebras $\alpha : X \to !X$ and $\beta : Y \to !Y$

then their tensor product is

$$X \otimes Y \xrightarrow{\alpha \otimes \beta} !X \otimes !Y \xrightarrow{m_{X,Y}} !(X \otimes Y)$$

It's routine to check that this really does give a coalgebra. The tensor unit in the category of coalgebras is $m_I : I \to !I$. (Notice that the diagrams for m_I above are exactly the conditions for it to be a coalgebra.) We also recall that a morphism from α to β is given by a function $f : X \to Y$ such that

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "f"'] & !X \arrow[d, "!f"] \\
Y \arrow[r, "\beta"'] & !Y
\end{tikzcd}$$

commutes.

We demand that this induced tensor is a categorical product. Sometimes it's convenient to express this requirement in a more concrete way. In order to do that, we need the notion of a comonoid object:

Definition 3.2. Let $\mathbb{C}$ be a symmetric monoidal category. A commutative comonoid in $\mathbb{C}$ is an object X equipped with a counit map $e : X \to I$ and a comultiplication $d : X \to X \otimes X$ such that the following commute:

$$\begin{tikzcd}
X \arrow[r, "d"] \arrow[dr, "\rho_X"'] & X \otimes X \arrow[d, "X \otimes e"] \\
 & X \otimes I
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "d"] \arrow[dr, "\lambda_X"'] & X \otimes X \arrow[d, "e \otimes X"] \\
 & X \otimes I
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "d"] \arrow[dr, "d"'] & X \otimes X \arrow[d, "\sigma_{X,X}"] \\
 & X \otimes X
\end{tikzcd}$$

$$\begin{tikzcd}
 & X \otimes X \arrow[r, "X \otimes d"] & X \otimes (X \otimes X) \\
X \arrow[ur, "d"] \arrow[dr, "d"'] & & \\
 & X \otimes X \arrow[r, "d \otimes X"'] & (X \otimes X) \otimes X \arrow[uu, "\alpha_{X,X,X}"']
\end{tikzcd}$$

(This is just the dual of the more familiar notion of a monoid object.)

The following is folklore; a version of it appears in Maneggia (2004, Prop. 1.15).

Proposition 3.3. *Let $\mathbb{C}$ be a symmetric monoidal category. Then the tensor unit is a terminal object and the tensor product is a categorical product iff there exist monoidal natural transformations with components*

$$e_X : X \to I$$

and

$$d_X : X \to X \otimes X$$

such that for every object X, (X, e_X, d_X) *is a commutative comonoid.*

Proof. Note that the statement of this proposition is implicitly appealing to the fact that the identity functor, the tensor functor and the constant functor at I are all lax monoidal in reasonably obvious ways.

It's helpful to make some of our conditions explicit. The naturality of e says that for every $f : X \to Y$,

$$\begin{tikzcd} X \arrow[dd, "f"'] \arrow[dr, "e_X"] & \\ & I \\ Y \arrow[ur, "e_Y"'] & \end{tikzcd} \tag{3.2}$$

and monoidality of e says that e_I is the identity and

$$\begin{tikzcd} & I \arrow[dd, "\lambda_I (= \rho_I)"] \\ X \otimes X \arrow[ur, "e_{X \otimes X}"] \arrow[dr, "e_X \otimes e_X"'] & \\ & I \otimes I \end{tikzcd} \tag{3.3}$$

commutes.

If I is terminal then we let e_X be the unique morphism from X to I, and clearly these conditions hold. Conversely if these conditions hold for some family of maps e_X, then taking $Y = I$ in (3.2) and using the fact that e_I is the identity, we conclude that any map $f : X \to I$ is equal to e_X, hence I is terminal. Notice that we haven't used (3.3), and that since λ_I is an isomorphism it must necessarily commute. Therefore (3.3) turns out to be redundant in the presence of the other conditions.

Turning now to d, the naturality condition says that for any map $f : X \to Y$,

$$\begin{tikzcd}
X \arrow[r, "d_X"] \arrow[d, "f"'] & X \otimes X \arrow[d, "f \otimes f"] \\
X \arrow[r, "d_Y"'] & Y \otimes Y
\end{tikzcd} \tag{3.4}$$

commutes; monoidality means that $d_I = \lambda_I$ (which is the same as ρ_I), and

$$\begin{tikzcd}
X \otimes Y \arrow[r, "1"] \arrow[d, "d_X \otimes d_X"'] & X \otimes Y \arrow[d, "d_{X \otimes Y}"] \\
(X \otimes X) \otimes (Y \otimes Y) \arrow[r, "\theta"'] & (X \otimes Y) \otimes (X \otimes Y)
\end{tikzcd} \tag{3.5}$$

commutes, where θ is the previously-mentioned combination of associativity and symmetry isomorphisms that associates the first X with the first Y and the second X with the second Y.

If $\otimes$ is a product, we let $d_X = \Delta_X$, the diagonal map on X. The conditions above, and the fact that (X, e_X, d_X) is a commutative comonoid, are then an easy consequence of the universal property of the product. Conversely if we have a family of maps d_X with these properties, we can show that $\otimes$ has the universal property of a product, as follows.

First we define the projection $\pi_1 : X \otimes Y \to X$ to be the composite

$$X \otimes Y \xrightarrow{X \otimes e_Y} X \otimes I \xrightarrow{\rho_X^{-1}} X$$

and similarly $\pi_2 : X \otimes Y \to Y$ is the composite

$$X \otimes Y \xrightarrow{e_X \otimes Y} I \otimes Y \xrightarrow{\lambda_Y^{-1}} Y$$

Now suppose we have an object A with $f : A \to X$ and $g : A \to Y$. We claim that the pairing $(f, g) : A \to X \otimes Y$ is the composite

$$A \xrightarrow{d_A} A \otimes A \xrightarrow{f \otimes g} X \otimes Y.$$

First we need to show that $\pi_1(f, g) = f$ and $\pi_2(f, g) = g$. These are similar, so we'll just do the first one. Explicitly we need to show that the composite

$$\begin{tikzcd}
A \arrow[r, "d_A"] & A \otimes A \arrow[r, "f \otimes g"] & X \otimes Y \arrow[r, "X \otimes e_Y"] & X \otimes I \arrow[r, "\rho_X^{-1}"] & X
\end{tikzcd}$$

is equal to f. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "d_A"] \arrow[d, "f"'] & A \otimes A \arrow[r, "f \otimes g"] \arrow[d, "f \otimes f"] & X \otimes Y \arrow[d, "X \otimes e_Y"] \\
X \arrow[r, "d_X"] \arrow[dr, "1"'] & X \otimes X \arrow[r, "X \otimes e_X"] & X \otimes I \arrow[dl, "\rho_X^{-1}"] \\
& X &
\end{tikzcd}$$

The left-hand square commutes by naturality of d, the right-hand square by naturality of e, and the lower triangle is the first of the conditions for (X, e_X, d_X) to be a comonoid.

Finally we need to show that this fill-in is unique, or equivalently that any $h : A \to X \otimes Y$ is equal to $(\pi_1 h, \pi_2 h)$. Suppose we have such a map h, and consider

$$\begin{tikzcd}
A \arrow[r, "d_A"] \arrow[d, "h"'] & A \otimes A \arrow[d, "h \otimes h"] & X \otimes Y \\
X \otimes Y \arrow[r, "d_{X \otimes Y}"'] \arrow[d, "1"'] & (X \otimes Y) \otimes (X \otimes Y) \arrow[r, "(X \otimes e_Y) \otimes (e_X \otimes Y)"] \arrow[d, "\theta"] & (X \otimes I) \otimes (I \otimes Y) \arrow[u, "\rho_X^{-1} \otimes \lambda_Y^{-1}"'] \\
X \otimes Y \arrow[r, "d_X \otimes d_Y"'] & (X \otimes X) \otimes (Y \otimes Y) \arrow[ur, "(X \otimes e_X) \otimes (e_Y \otimes Y)"'] &
\end{tikzcd}$$

The upper square is (3.4), the lower left square is (3.5), and the triangle commutes because I is terminal. Then since (X, e_X, d_X) and (Y, e_Y, d_Y) are commutative comonoids, the edge path from the bottom left to the top right composes to the identity on $X \otimes Y$. The edge path from top left to top right is the pairing $(\pi_1 h, \pi_2 h)$, which since the diagram commutes is therefore equal to h. ☐

So now instead of demanding that the induced tensor on the category of coalgebras be a product, we could equivalently ask for a pair of monoidal natural transformations as above. This is getting more explicit, but still involves the category of coalgebras overtly. We are aiming to express the conditions purely in terms of structure that exists in the category $\mathbb{C}$ itself, which can be done as follows:

Proposition 3.4. *Let $\mathbb{C}$ be a symmetric monoidal closed category equipped with a monoidal comonad $(!, \varepsilon, \delta)$. The induced tensor on the category of $!$-coalgebras is a product (and its unit is a terminal object) iff the following structure exists on $\mathbb{C}$:*

- *A monoidal natural transformation e from $!$ to the constant functor I; and a monoidal natural transformation d from $!$ to $\lambda X.!X \otimes !X$. For these transformations to be monoidal means that $e_I m_I$ is the identity and the following three diagrams commute:*

$$\begin{tikzcd}
!X \otimes !Y \arrow[r, "m_{X,Y}"] \arrow[d, "e_X \otimes e_Y"'] & !(X \otimes Y) \arrow[d, "e_{X \otimes Y}"] \\
I \otimes I \arrow[r, "\lambda_I"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "m_I"] \arrow[d, "\lambda_I"'] & !I \arrow[d, "d_I"] \\
I \otimes I \arrow[r, "m_I \otimes m_I"'] & !I \otimes !I
\end{tikzcd}$$

$$\begin{tikzcd}
!X \otimes !Y \arrow[r, "d_X \otimes d_Y"] \arrow[dd, "m_{X,Y}"'] & (!X \otimes !X) \otimes (!Y \otimes !Y) \arrow[d, "\theta"] \\
 & (!X \otimes !Y) \otimes (!X \otimes !Y) \arrow[d, "m_{X,Y} \otimes m_{X,Y}"] \\
!(X \otimes Y) \arrow[r, "d_{X \otimes Y}"'] & !(X \otimes Y) \otimes !(X \otimes Y)
\end{tikzcd}$$

- *For every object X, $(!X, e_X, d_X)$ is a commutative comonoid*

- *and for every object* X*, the following four diagrams commute:*

$$\begin{tikzcd}
!X \arrow[dr, "e_X"] \arrow[dd, "\delta_X"'] & \\
& I \\
!!X \arrow[ur, "e_{!X}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
!X \arrow[r, "d_X"] \arrow[d, "\delta_X"'] & !X \otimes !X \arrow[d, "\delta_X \otimes \delta_X"] \\
!!X \arrow[r, "d_{!X}"'] & !!X \otimes !!X
\end{tikzcd}$$

$$\begin{tikzcd}
!X \arrow[r, "e_X"] \arrow[d, "\delta_X"'] & I \arrow[d, "m_I"] \\
!!X \arrow[r, "!e_X"'] & !I
\end{tikzcd}
\qquad
\begin{tikzcd}
!X \arrow[r, "d_X"] \arrow[ddd, "\delta_X"'] & !X \otimes !X \arrow[d, "\delta_X \otimes \delta_X"] \\
& !!X \otimes !!X \arrow[dd, "m_{!X,!X}"] \\
& \\
!!X \arrow[r, "!d_X"'] & !(!X \otimes !X)
\end{tikzcd}$$

(The top two diagrams say that δ_X *preserves the comonoid structure, while the lower two ensure that* e_X *and* d_X *give rise to coalgebra morphisms.)*

Proof sketch. Its reasonably easy to see that any linear exponential comonad gives rise to the structure above; the hard part is the converse. Given any coalgebra $\alpha : X \to !X$ we define a comonoid structure on it, which we'll denote with the bold letters **e** and **d** to distinguish it from the transformations e and d whose existence we assume. The counit component $\mathbf{e}_X$ is the composite

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & !X \arrow[d, "!\alpha"] \\
!X \arrow[r, "\delta_X"] \arrow[d, "e_X"'] & !!X \arrow[d, "!e_X"] \\
I \arrow[r, "m_I"'] & !I
\end{tikzcd}$$

(The top square commutes because α is a coalgebra, and the bottom square is one of our conditions.) The comultiplication component $\mathbf{d}_X$ is

$$\begin{tikzcd}
X \arrow[rr, "\alpha"] \arrow[d, "\alpha"'] & & !X \arrow[d, "!\alpha"] \\
!X \arrow[rr, "\delta_X"'] \arrow[d, "d_X"'] & & !!X \arrow[d, "!d_X"] \\
!X \otimes !X \arrow[r, "\delta_X \otimes \delta_X"'] \arrow[d, "\varepsilon_X \otimes \varepsilon_X"'] \arrow[dr, "1"] \arrow[dr, phantom, "\text{⊹}"'] & !!X \otimes !!X \arrow[r, "m_{!X,!X}"'] \arrow[d, "!\varepsilon_X \otimes !\varepsilon_X"] & !(!X \otimes !X) \arrow[d, "!(\varepsilon_X \otimes \varepsilon_X)"] \\
X \otimes X \arrow[r, "\alpha \otimes \alpha"'] & !X \otimes !X \arrow[r, "m_{X,X}"'] & !(X \otimes X)
\end{tikzcd}$$

All the cells here commute for simple reasons, with the exception of the triangle at lower left (which we have marked with the puncture symbol ⊹ to indicate that it doesn't commute). The reason the outer rectangle commutes nonetheless is that, although the punctured triangle does not commute per se, when composed with $d_X\alpha$ it does. To see that, consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & !X \arrow[d, "\delta_X"'] \arrow[ddr, "d_X"] & \\
!X \arrow[r, "!\alpha"] \arrow[d, "d_X"'] & !!X \arrow[d, "d_{!X}"] & \\
!X \otimes !X \arrow[r, "!\alpha \otimes !\alpha"] \arrow[d, "\varepsilon_X \otimes \varepsilon_X"'] & !!X \otimes !!X \arrow[d, "\varepsilon_{!X} \otimes \varepsilon_{!X}"'] & !X \otimes !X \arrow[l, "\delta_X \otimes \delta_X"] \arrow[dl, "1"] \\
X \otimes X \arrow[r, "\alpha \otimes \alpha"'] & !X \otimes !X &
\end{tikzcd}$$

Here all the cells commute for simple reasons, so it follows that our comultiplication is indeed a coalgebra morphism.

We need to check that **e** and **d** are natural in the category of coalgebras, so let $\alpha : X \to\, !X$ and $\beta : Y \to\, !Y$ be coalgebras and let $f : X \to Y$ be a morphism between them. Now consider the diagrams

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "\alpha"'] & & Y \arrow[d, "\beta"] \\
!X \arrow[rr, "!f"] \arrow[dr, "e_X"'] & & !Y \arrow[dl, "e_Y"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\
!X \arrow[r, "!f"] \arrow[d, "d_X"'] & !Y \arrow[d, "d_Y"] \\
!X \otimes !X \arrow[r, "!f \otimes !f"] \arrow[d, "\varepsilon_X \otimes \varepsilon_X"'] & !Y \otimes !Y \arrow[d, "\varepsilon_Y \otimes \varepsilon_Y"] \\
X \otimes X \arrow[r, "f \otimes f"'] & Y \otimes Y
\end{tikzcd}$$

The top cells commute because f is a coalgebra morphism and the others by naturality, so we have indeed defined natural transformations.

We also need to check that our natural transformations are monoidal. Recall from the proof of Prop. 3.3 that the binary case for **e** is redundant, so there are three cases to check. Firstly $\mathbf{e}_I$ must be the identity; but $\mathbf{e}_I = e_I m_I$, which is the identity since e is monoidal. Secondly $\mathbf{d}_I$ should equal λ_I, so consider

$$\begin{tikzcd}
I \arrow[r, "\lambda_I"] \arrow[d, "m_I"'] & I \otimes I \arrow[d, "m_I \otimes m_I"'] \arrow[dr, "1"] & \\
!I \arrow[r, "d_I"'] & !I \otimes !I \arrow[r, "\varepsilon_I \otimes \varepsilon_I"'] & I \otimes I
\end{tikzcd}$$

The square commutes because d is monoidal, and the triangle because ε is. For the binary case, consider the following.

$$\begin{tikzcd}
X \otimes Y \arrow[rr, "\alpha \otimes \beta"] \arrow[d, "\alpha \otimes \beta"'] & & !X \otimes !Y \arrow[d, "m_{X,Y}"] \\
!X \otimes !Y \arrow[rr, "m_{X,Y}"] \arrow[d, "d_X \otimes d_Y"'] & & !(X \otimes Y) \arrow[d, "d_{X \otimes Y}"] \\
(!X \otimes !X) \otimes (!Y \otimes !Y) \arrow[r, "\theta"'] \arrow[d, "(\varepsilon_X \otimes \varepsilon_X) \otimes (\varepsilon_Y \otimes \varepsilon_Y)"] & (!X \otimes !Y) \otimes (!X \otimes !Y) \arrow[r, "m_{X,Y} \otimes m_{X,Y}"'] \arrow[dr, "(\varepsilon_X \otimes \varepsilon_Y) \otimes (\varepsilon_X \otimes \varepsilon_Y)"] & !(X \otimes Y) \otimes !(X \otimes Y) \arrow[d, "\varepsilon_{X \otimes Y} \otimes \varepsilon_{X \otimes Y}"] \\
(X \otimes X) \otimes (Y \otimes Y) \arrow[rr, "\theta"'] & & (X \otimes Y) \otimes (X \otimes Y)
\end{tikzcd}$$

The middle rectangle commutes by monoidality of d, the triangle by monoidality of ε and the trapezium by naturality of θ.

The one remaining task to check that this does indeed define a comonoid. We omit this part; there's a proof in Maneggia (2004, Prop. 1.13). ☐

In terms of linear logic, the maps d_X model contraction and the maps e_X model weakening, so they're often referred to as *contraction maps* and *weakening maps* respectively.

3.2 Partial Monoidal Categories

In this section we develop the theory of partial monoidal categories. It is presented here without making any serious use of 2-category theory: the occasional remarks about 2-monads and the like may be safely ignored. On the other hand, we do need a little of the theory of enriched categories: everything we need (and much more) can be found in the first chapter of Kelly (1982).

Intuitively, a partial monoidal category is a category whose tensor product is only partially defined. In order to formalise this simple idea, we recall the notion of a zero object, and describe how to add one to an existing category.

3.2.1 Zero Objects

A zero object is simply an initial object which is also terminal. Suppose we have a category $\mathbb{C}$ with a zero object Z. Then for any object X there is a unique morphism

$$\begin{tikzcd} X \arrow[r, "z_{X,Z}"] & Z \end{tikzcd}$$

and another unique morphism

$$\begin{tikzcd} Z \arrow[r, "z_{Z,X}"] & X \end{tikzcd}$$

Thus for any two objects X and Y, we can form the (unique) composite $z_{X,Y} = z_{Z,Y}z_{X,Z}$:

$$\begin{tikzcd} X \arrow[r, "z_{X,Z}"] & Z \arrow[r, "z_{Z,Y}"] & Y \end{tikzcd}$$

In other words, between any two objects there is a unique 'zero morphism', and furthermore (because of the uniqueness) any composite involving a zero morphism is another zero morphism. For example, for any arrow $f : A \to B$ we have $z_{B,C}f = z_{A,C}$.

Example 3.5. In the category Rel of sets and relations, the empty set is a zero object. A zero morphism is an empty relation.

Example 3.6. In the category Mon of monoids and homomorphisms, the trivial monoid is a zero object. A zero morphism is one that sends every element of the source monoid to the unit element of the target.

A category may have more than one zero object, but they will all be isomorphic. In most of our examples there is actually only one; otherwise we shall insist that one of them be singled out as 'the' zero object, in which case we say we have a 'category with selected zero object'.

3.2.2 Zero morphisms and enrichment

The property of having zero morphisms can be formulated abstractly in terms of categorical enrichment (Kelly, 1982). We enrich over the category Set_* of pointed sets, in which an object is a set with a selected base point $*$ and a morphism is a function that preserves the base point. (This is the category of algebras for the monad $\lambda X.1 + X$ on Set. It happens to be equivalent to the Kleisli category – i.e. every algebra is isomorphic to a free algebra – which is the category Pfn of sets and partial functions.)

The category Set_* has rather a lot of structure. It has coproducts given by coalesced sum (where the base points are identified), and also products given by the ordinary cartesian product (with $(*, *)$ as base point). It even has a zero object: the one-element set $\{*\}$. More to the point, Set_* is symmetric monoidal closed. The tensor product $X \otimes Y$ is a smash product, meaning the set

$$\{(x, y) \in X \times Y \mid x = * \iff y = *\}$$

with base point $(*, *)$. Thus a base-preserving function from $X \otimes Y$ to A is a function of two arguments that preserves the base point in each argument.[2] The tensor unit is the set $\{*, i\}$ with one non-base point i. The linear function space $X \multimap Y$ is the set of all base-preserving functions $f : X \to Y$; the base point of this set is the function that maps each $x \in X$ to $* \in Y$.

A Set_*-enriched category is thus a category in which each hom-set has a distinguished base point (i.e. zero morphism), such that composition preserves the base point in both arguments. It need not actually have a zero object, but any initial or terminal object must by definition have a unique endomorphism and so its identity will be the zero map, hence it will in fact be a zero object.

Since Set_* is equivalent to Pfn, we can alternatively think of a Set_*-enriched category as a generalisation of an ordinary (i.e. Set-enriched) category in which the composition operations are allowed to be partial. Given two composable morphisms

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

there may or may not be a composite gf.

Clearly there is a 2-category $\mathrm{Set}_*\mathrm{Cat}$ whose objects are the Set_* categories, whose 1-cells are Set_*-enriched functors and whose 2-cells are natural transformations.

---

[2] This is analogous to the way that in the category of commutative monoids a morphism from $X \otimes Y$ is a bilinear homomorphism.

3.2.3 Adding a zero object

Given an existing category, we may add a zero object freely. It's convenient to do it in two stages, first adding the zero morphisms and then the actual zero object. The linear element functor $\lambda X.\mathrm{Set}_*(I, X)$ from Set_* to Set – which is simply the forgetful functor that forgets which element of a set is the base point – has a left adjoint $\lambda X.1 + X$. It follows that the 'impoverishing' 2-functor from Set_*-Cat to Cat has a left 2-adjoint M that freely enriches each homset. (Kelly (1982) has the details of this construction.)

In concrete terms, the 2-functor M adds a new zero morphism to each homset and leaves everything else untouched. Now let ZCat be the 2-category whose objects are categories with selected zero object (which we call *Z-categories*), and whose morphisms are functors that preserve the (selected) zero object (called *Z-functors*). Note that a Z-functor will also preserve zero morphisms. The forgetful 2-functor $V : \mathrm{ZCat} \to \mathrm{Set}_*\text{-Cat}$ (which forgets about the selected zero object) has a left 2-adjoint N that adds a new zero object (and selects it). Thus we can compose these 2-adjunctions

$$\begin{tikzcd}
\mathrm{Cat} \arrow[r, shift left, "M"] & \mathrm{Set}_*\text{-Cat} \arrow[l, shift left, "U"] \arrow[r, shift left, "N"] & \mathrm{ZCat} \arrow[l, shift left, "V"]
\end{tikzcd}$$

and we'll write Zero for the composite NM, the *free zero-object completion*.

Clearly there is a canonical embedding of $\mathbb{C}$ into $\mathrm{Zero}(\mathbb{C})$, and also there's a natural functor from $\mathrm{Zero}(\mathrm{Zero}(\mathbb{C}))$ to $\mathrm{Zero}(\mathbb{C})$ that identifies the two new zero objects. These operations constitute the unit and multiplication respectively of the 2-monad defined by this 2-adjunction.

3.2.4 Monoidal Structure

Adding a zero object disturbs the additive structure: for example, even if $\mathbb{C}$ has finite products it isn't necessarily the case that $\mathrm{Zero}(\mathbb{C})$ will. However the multiplicative structure can be lifted from $\mathbb{C}$ to $\mathrm{Zero}(\mathbb{C})$.

The underlying reason for this is that M and N are monoidal functors with respect to the natural monoidal structure on the 2-categories that they connect. However it seems complicated to define the 2-categorical notions that are needed to make this precise, so we shall confine ourselves to concrete cases rather than invoking the general theory.

Definition 3.7. A (symmetric) *Z-monoidal* category is a (symmetric) monoidal category $\mathbb{C}$ with selected zero object whose tensor product functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ preserves the selected zero object in both arguments. A *symmetric Z-monoidal closed* category also has a functor $\multimap : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}$ that preserves the zero object in both arguments, such that for every object X we have $- \otimes X \dashv X \multimap -$. A *Z-$*$-autonomous* category is a symmetric Z-monoidal closed category $\mathbb{C}$ with a selected object $\perp$ such that for every object X the canonical map

$$\zeta_X^{\perp} : X \to (X \multimap \perp) \multimap \perp,$$

defined as the exponential transpose of the reversed evaluation map

$$\varepsilon_{\perp}^{X} \sigma_{X \multimap \perp, X} : X \otimes (X \multimap \perp) \to \perp,$$

is an isomorphism.

Abstractly, a Z-monoidal category is a pseudomonoid in ZCat.

Definition 3.8. A *partial monoidal* category is a category $\mathbb{C}$ together with a Z-monoidal structure on $\mathrm{Zero}(\mathbb{C})$. Similarly a *partial $*$-autonomous* category is a category $\mathbb{C}$ with a Z-$*$-autonomous structure on $\mathrm{Zero}(\mathbb{C})$ etc.

Abstractly a partial monoidal category is a pseudomonoid in the Kleisli 2-category of the Zero 2-monad. One may regard a partial monoidal category as a monoidal category whose tensor product is only partially defined. Every monoidal category is partial monoidal in the obvious way, i.e. any tensor product on $\mathbb{C}$ can be lifted to $\mathrm{Zero}(\mathbb{C})$. We can lift the closed or $*$-autonomous structure in a similar way.

3.3 Disjunctive Dualisation

A *self-dual category* is a category $\mathbb{C}$ equipped with a contravariant endofunctor $\neg$ such that $\neg\neg$ is the identity. If $\mathbb{C}$ and $\mathbb{D}$ are self-dual categories, a duality-preserving functor between them is a functor

$F : \mathbb{C} \to \mathbb{D}$ such that for every object X we have $F(\neg X) = \neg FX$ and for every arrow f we have $F(\neg f) = \neg Ff$. Let DualCat be the 2-category whose objects are self-dual categories, whose 1-cells are duality-preserving functors and whose 2-cells are natural transformations. There's an obvious forgetful functor $F : \mathrm{DualCat} \to \mathrm{Cat}$, which has both a left adjoint L and a right adjoint R.

The effect of R on categories is to take the category $\mathbb{C}$ to the self-dual category $\mathbb{C} \times \mathbb{C}^{op}$ with $\neg(X, Y) = (Y, X)$ for X, Y either objects or morphisms in $\mathbb{C}$.

The other adjoint has $L(\mathbb{C}) = \mathbb{C} + \mathbb{C}^{op}$, in which an object is X^+ or X^- for some object $X \in \mathbb{C}$ and a morphism is $f^+ : X^+ \to Y^+$ or $f^- : Y^- \to X^-$ for some $f : X \to Y$ in $\mathbb{C}$. Here $\neg X^+ = X^-$ and $\neg X^- = X^+$.

If $\mathbb{C}$ is a symmetric monoidal closed category with finite products, the category $R(\mathbb{C}) = \mathbb{C} \times \mathbb{C}^{op}$ is $*$-autonomous. This construction is discussed by Hyland and Schalk (2003) under the name 'simple dualisation'. The directly analogous result does not hold for the left adjoint L, but we are now in a position to state a weaker result that does.

Proposition 3.9. *If $\mathbb{C}$ is (partial) symmetric monoidal closed then $\mathbb{C} + \mathbb{C}^{op}$ is partial $*$-autonomous.*

The rest of this section is devoted to proving Prop. 3.9. Suppose $\mathbb{C}$ is a symmetric monoidal closed category with unit I. The objects of $\mathbb{C} + \mathbb{C}^{op}$ are of two types, which we'll write as:

- X^+, where $X \in \mathbb{C}$,
- X^-, where $X \in \mathbb{C}$,

and the morphisms are:

- $f^+ : X^+ \to Y^+$, where $f : X \to Y$ in $\mathbb{C}$,
- $f^- : Y^- \to X^-$, where $f : X \to Y$ in $\mathbb{C}$.

We refer to the two classes of object and morphism as 'positive' or 'negative', following the superscript. If we want a variable that could be of either type we write it as $X^\pm$ etc; and adopt the convention that a variable with no superscript refers to an object or morphism of $\mathbb{C}$.

The tensor product is defined by cases:

$$\begin{aligned} X^+ \otimes Y^+ &= (X \otimes Y)^+ \\ X^+ \otimes Y^- &= (X \multimap Y)^- \\ X^- \otimes Y^+ &= (Y \multimap X)^- \end{aligned}$$

and the tensor unit is I^+. Notice that we have not defined a tensor product for the case $X^- \otimes Y^-$, so the monoidal structure is truly partial. The definition on morphisms follows the same pattern, e.g. $f^+ \otimes g^- = (f \multimap g)^-$.

The structural isomorphisms are defined in the obvious way. For example $\sigma_{X^+,Y^+} = (\sigma_{X,Y})^+$ and $\sigma_{X^+,Y^-} = (1_{X \multimap Y})^-$; similarly α_{W^+,X^+,Y^-} is $(\psi_{W,X,Y})^-$ where

$$\psi_{W,X,Y} : W \multimap (X \multimap Y) \to (W \otimes X) \multimap Y$$

is the canonical such isomorphism in $\mathbb{C}$.

The linear function space is defined as:

$$\begin{aligned} X^+ \multimap Y^+ &= (X \multimap Y)^+ \\ X^- \multimap Y^- &= (Y \multimap X)^+ \\ X^+ \multimap Y^- &= (Y \otimes X)^- \end{aligned}$$

This time the case $X^- \multimap Y^+$ has been left undefined.

We claim that these data describe a closed category, so we must establish that for each object $X^\pm$ there is an adjunction

$$(- \otimes X^\pm) \dashv (X^\pm \multimap -)$$

in $\mathrm{Zero}(\mathbb{C} + \mathbb{C}^{\mathrm{op}})$. We shall describe this adjunction by giving its unit and counit, and showing that they satisfy the necessary conditions. Suppose that the corresponding adjunction in $\mathbb{C}$ is given with unit η^X, having components

$$\eta^X_Y : Y \to X \multimap (Y \otimes X)$$

and counit ε^X (the evaluation map) having components

$$\varepsilon^X_Y : (X \multimap Y) \otimes X \to Y.$$

In order to define our adjunction we need one additional ingredient:

define

$$\zeta_Y^X : Y \to (Y \multimap X) \multimap X$$

that will be the exponential transpose of the ‘reversed evaluation’ map

$$Y \otimes (Y \multimap X) \xrightarrow{\sigma_{Y,Y\multimap X}} (Y \multimap X) \otimes X \xrightarrow{\varepsilon_X^Y} X;$$

in terms of the unit and counit that is

$$\zeta_Y^X = \big((Y \multimap X) \multimap (\varepsilon_X^Y . \sigma_{Y,Y\multimap X})\big)\eta_Y^{Y\multimap X}. \tag{3.6}$$

In the dualised category we define natural transformations $\varepsilon^{X^\pm}$ and $\eta^{X^\pm}$ to be the counit and unit of the adjunction. (These are notationally distinguished from the transformations in $\mathbb{C}$ by the ± superscripts.) The definitions are:

$$\begin{array}{ll}
\varepsilon_{Y^+}^{X^+} = (\varepsilon_Y^X)^+ & \qquad \eta_{Y^+}^{X^+} = (\eta_Y^X)^+ \\
\varepsilon_{Y^-}^{X^+} = (\eta_Y^X)^- & \qquad \eta_{Y^-}^{X^+} = (\varepsilon_Y^X)^- \\
\varepsilon_{Y^+}^{X^-} = z_{Z,Y^-} & \qquad \eta_{Y^+}^{X^-} = (\zeta_Y^X)^+ \\
\varepsilon_{Y^-}^{X^-} = (\zeta_Y^X)^- & \qquad \eta_{Y^-}^{X^-} = z_{Y^-,Z}
\end{array}$$

It is not instantly clear that these morphisms even have the right type, but it’s easy to check. For example, we need

$$\varepsilon_{Y^-}^{X^-} : (X^- \multimap Y^-) \otimes X^- \to Y^-,$$

and the left hand side is

$$\begin{aligned}
(X^- \multimap Y^-) \otimes X^- &= (Y \multimap X)^+ \otimes X^- \\
&= ((Y \multimap X) \multimap X)^-
\end{aligned}$$

so our morphism must be f^- for some $f : Y \to (Y \multimap X) \multimap X$.

To see that these transformations are natural, observe that it suffices for each case to be natural separately since every non-zero morphism connects two objects with the same sign. (A naturality square that has a zero morphism on each of two parallel sides must of course commute.)

We will need the following lemma.[3]

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[3] The reader who is familiar with the proof-net characterisation of free symmetric monoidal closed categories will see right away that it must be true, since the free symmetric monoidal closed category generated by X and Y has a unique endomorphism on $Y \multimap X$.

Lemma 3.10. *For any two objects* X *and* Y*, the composite*

$$Y \multimap X \xrightarrow{\zeta^X_{Y\multimap X}} ((Y \multimap X) \multimap X) \multimap X \xrightarrow{\zeta^X_Y \multimap X} Y \multimap X$$

is the identity on $Y \multimap X$.

Proof. For any objects A, B, C, we write

$$\mathrm{curry} : \mathbb{C}(A \otimes B, C) \to \mathbb{C}(A, B \multimap C)$$

and its inverse

$$\mathrm{uncurry} : \mathbb{C}(A, B \multimap C) \to \mathbb{C}(A \otimes B, C)$$

for the natural isomorphisms between homsets determined by the adjunction. In these terms, we have the identities

$$\zeta^X_Y = \mathrm{curry}(\varepsilon^Y_X.\sigma_{Y,Y\multimap X})$$

and

$$\varepsilon^X_Y = \mathrm{uncurry}(1_{X\multimap Y}),$$

which we shall use later.

As a first step to our result, we note the general fact that given any composite

$$A \xrightarrow{g} B \multimap C \xrightarrow{f\multimap C} B' \multimap C$$

for some $f : B' \to B$, we have

$$\mathrm{uncurry}((f \multimap X).g) = \mathrm{uncurry}(g).(A \otimes f).$$

To see this, consider the following diagram in $[\mathbb{C}^{\mathrm{op}} \times \mathbb{C}^{\mathrm{op}} \times \mathbb{C}, \mathrm{Set}]$:

$$\begin{tikzcd}
\mathbb{C}(A \otimes B, C) \arrow[d, "-.(A\otimes f)"'] & \mathbb{C}(A, B \multimap C) \arrow[l, "\mathrm{uncurry}"'] \arrow[d, "(f \multimap C).-"] \\
\mathbb{C}(A \otimes B', C) & \mathbb{C}(A, B' \multimap C) \arrow[l, "\mathrm{uncurry}"]
\end{tikzcd}$$

which commutes by naturality of uncurry. Now compare the two different ways to chase g from the top right cell to the bottom left.

We can use this to rework our claim into an equivalent form that will be easier to attack:

$$\begin{aligned}
& (\zeta^X_Y \multimap X).\zeta^X_{Y\multimap X} = 1_{Y\multimap X} \\
\iff \quad & \text{uncurry}((\zeta^X_Y \multimap X).\zeta^X_{Y\multimap X}) = \text{uncurry}(1_{Y\multimap X}) \\
\iff \quad & \text{uncurry}(\zeta^X_{Y\multimap X}).((Y \multimap X) \otimes \zeta^X_Y) = \varepsilon^Y_X \\
\iff \quad & \varepsilon^{Y\multimap X}_X.\sigma_{Y\multimap X,(Y\multimap X)\multimap X}.((Y \multimap X) \otimes \zeta^X_Y) = \varepsilon^Y_X
\end{aligned}$$

In diagrammatic terms, we require the following to commute, where for clarity we abbreviate $Y' = Y \multimap X$ and $Y'' = Y' \multimap X$.

$$\begin{tikzcd}
Y' \otimes Y'' \arrow[r, "\sigma_{Y',Y''}"] & Y'' \otimes Y' \arrow[d, "\varepsilon^{Y'}_X"] \\
Y' \otimes Y \arrow[u, "Y' \otimes \zeta^X_Y"] \arrow[r, "\varepsilon^Y_X"'] & X
\end{tikzcd}$$

If we permute the arrows that meet at the top left (i.e. using the naturality of σ) and invert the isomorphism $\sigma_{Y',Y}$, we get the equivalent diagram

$$\begin{tikzcd}
Y \otimes Y' \arrow[r, "\zeta^X_Y \otimes Y'"] \arrow[d, "\sigma_{Y,Y'}"'] & Y'' \otimes Y' \arrow[d, "\varepsilon^{Y'}_X"] \\
Y' \otimes Y \arrow[r, "\varepsilon^Y_X"'] & X
\end{tikzcd}$$

To see why this diagram commutes, consider the naturality diagram below, and chase $1_{Y''}$ from the top right to the bottom left in each of the two possible ways.

$$\begin{tikzcd}
\mathbb{C}(Y'' \otimes Y', X) \arrow[d, "-.(\zeta^X_Y \otimes Y')"'] & \mathbb{C}(Y'', Y'') \arrow[l, "\text{uncurry}"'] \arrow[d, "-.\zeta^X_Y"] \\
\mathbb{C}(Y \otimes Y', X) & \mathbb{C}(Y, Y'') \arrow[l, "\text{uncurry}"]
\end{tikzcd}$$

This relies on the facts we noted earlier, that $\text{uncurry}(\zeta^X_Y)$ is equal to $\varepsilon^Y_X.\sigma_{Y,Y'}$ and $\text{uncurry}(1_{Y'\multimap X})$ is equal to $\varepsilon^{Y'}_X$. □

To confirm that our natural transformations ε and η do indeed define an adjunction, we need to check that the composites

$$X^{\pm} \multimap Y^{\pm} \xrightarrow{\eta^{X^{\pm}}_{X^{\pm} \multimap Y^{\pm}}} X^{\pm} \multimap \big((X^{\pm} \multimap Y^{\pm}) \otimes X^{\pm}\big) \xrightarrow{X^{\pm} \multimap \varepsilon^{X^{\pm}}_{Y^{\pm}}} X^{\pm} \multimap Y^{\pm}$$

and

$$Y^{\pm} \otimes X^{\pm} \xrightarrow{\eta^{X^{\pm}}_{Y^{\pm}} \otimes X^{\pm}} \big(X^{\pm} \multimap (Y^{\pm} \otimes X^{\pm})\big) \otimes X^{\pm} \xrightarrow{\varepsilon^{X^{\pm}}_{Y^{\pm} \otimes X^{\pm}}} Y^{\pm} \otimes X^{\pm}$$

are both identities for every two objects $X^{\pm}$ and $Y^{\pm}$. There are four cases according to the signs of $X^{\pm}$ and $Y^{\pm}$. In two cases the diagrams reduce to the equivalent two diagrams in $\mathbb{C}$. The third case is trivial, since the identity on Z is the zero morphism, and the fourth case is Lemma 3.10.

For the $*$-autonomy we set $\bot = I^{-}$. Then we have $X^{+} \multimap \bot = X^{-}$ and $X^{-} \multimap \bot = X^{+}$ (and the same for morphisms), so

$$(X^{\pm} \multimap \bot) \multimap \bot = X^{\pm}.$$

We can calculate $\zeta^{X^{\pm}}_{Y^{\pm}}$ using formula (3.6), and we find that

$$\begin{aligned} \zeta^{X^{+}}_{Y^{+}} &= (\zeta^{X}_{Y})^{+} \\ \zeta^{X^{+}}_{Y^{-}} &= z_{Z,Y^{-}} \\ \zeta^{X^{-}}_{Y^{+}} &= (\eta^{X}_{Y})^{+} \\ \zeta^{X^{-}}_{Y^{-}} &= (\varepsilon^{X}_{Y})^{-}, \end{aligned}$$

which relates to the definitions of $\varepsilon^{X^{\pm}}_{Y^{\pm}}$ and $\eta^{X^{\pm}}_{Y^{\pm}}$ in a pleasantly symmetrical way.

Our one remaining obligation is to show that our category is $*$-autonomous, i.e. that $\zeta^{I^{-}}_{Y^{\pm}}$ is always an isomorphism. By our description of ζ above, it suffices to show that η^{I}_{X} and ε^{I}_{X} are both isomorphisms in $\mathbb{C}$. This is fairly standard:

Lemma 3.11. *In a symmetric monoidal closed category* $\mathbb{C}$ *with tensor unit* I *and residuating adjunctions* $(\eta^{X}, \varepsilon^{X})$*, the arrows* η^{I}_{X} *and* ε^{I}_{X} *are both isomorphisms for each object* X*.*

Proof. In $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$, we have the following natural isomorphism (with parameters A and X):

$$\mathbb{C}(A, X) \cong \mathbb{C}(A \otimes I, X) \cong \mathbb{C}(A, I \multimap X) \tag{3.7}$$

The first isomorphism represents composition with $\rho_A\colon A \otimes I \longrightarrow A$, and the second is from the closure adjunction. Using the Yoneda lemma we conclude that $X \cong I \multimap X$, and that this isomorphism is given by the arrow that results from taking $\mathrm{id}_X \in \mathbb{C}(A, X)$ and chasing it rightwards. In other words,

$$\begin{tikzcd}
X \arrow[r, "\eta^I_X"'] & I \multimap (X \otimes I) \arrow[r, "I \multimap \rho_X"'] & I \multimap X
\end{tikzcd}$$

is an isomorphism. We know that $I \multimap \rho_X$ is an isomorphism, hence η^I_X must be one as well.

For ε^I_X we use the same equation (3.7) in the other direction. Taking $\mathrm{id}_{I \multimap X}$ in $\mathbb{C}(A, I \multimap X)$ and chasing leftwards, we get the isomorphism

$$\begin{tikzcd}
I \multimap X \arrow[r, "\rho^{-1}_{I \multimap X}"'] & (I \multimap X) \otimes I \arrow[r, "\varepsilon^I_X"'] & X
\end{tikzcd}$$

and since $\rho^{-1}_{I \multimap X}$ is an isomorphism, ε^I_X must be one also. □

It follows then that $\mathbb{C} + \mathbb{C}^{\mathrm{op}}$ is indeed $*$-autonomous.

3.3.1 The Hypercoherence Completion

This dualisation construction is a rich source of interesting non-trivial partial $*$-autonomous categories, but the principal reason for our interest here is that the hypercoherence completions – described below – is just as effective when the starting category is partial $*$-autonomous as when it is total, and gives rise to a completed categories that is (total) $*$-autonomous.[4] As promised in the introduction, we shall see at the end of this section that Melliès's category of hypergraphs is $\mathrm{HCoh}(\mathbf{1} + \mathbf{1}^{\mathrm{op}})$ where $\mathbf{1}$ is the terminal category and we equip $\mathbf{1} + \mathbf{1}^{\mathrm{op}}$ with the partial $*$-autonomous structure described above.

We briefly describe the category of hypercoherences (Ehrhard, 1993). (Recall that the category of hypergraphs has already been defined in §2.5.)

Definition 3.12. A hypercoherence X is a set $|X|$ whose elements are called *atoms* together with an *atomic coherence* $\Gamma(X) \subseteq \mathcal{P}^+_{\mathrm{fin}}(|X|)$

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[4] The zero-object completion gives a trivial example of this phenomenon: any partial structure on $\mathbb{C}$ lifts to the corresponding total structure on $\mathrm{Zero}(\mathbb{C})$, by definition.

(i.e. the atomic coherence is a set of non-empty finite subsets of $|X|$) subject to the restriction that for every $x \in |X|$ the singleton $\{x\}$ is in $\Gamma(X)$. Say that a set of atoms is (*atomically*) *coherent* if it's in $\Gamma(X)$.

Where it's unlikely to create confusion, we often omit the bars and simply write X to mean the underlying set of the hypercoherence X.

The *strict atomic coherence* $\Gamma^*(X)$ is the set

$$\Gamma^*(X) = \{S \in \Gamma(X) \mid |S| > 1\},$$

and its elements are called 'strictly (atomically) coherent'. A set of atoms is *incoherent* just when it's not strictly coherent; so the singletons (and only the singletons) are both coherent and incoherent.

A *state* or *clique* is a set of atoms whose every non-empty finite subset is atomically coherent. Similarly an *anticlique* is a set of atoms whose every non-empty subset is incoherent.

Given hypercoherences X and Y, a hypercoherence morphism $f : X \to Y$ is a relation between $|X|$ and $|Y|$ such that for every $S \subseteq_{\text{fin}} f$,

$$\pi_1[S] \in \Gamma(X) \;\Rightarrow\; \pi_2[S] \in \Gamma(Y)$$

and

$$\pi_1[S] \in \Gamma^*(X) \;\Rightarrow\; \pi_2[S] \in \Gamma^*(Y).$$

(Note that this second condition is equivalent to asking that $\pi_2[S]$ incoherent implies $\pi_1[S]$ incoherent.)

The tensor unit is $I = \{i\}$, and we let $|X \otimes Y| = |X| \times |Y|$, with $S \in \Gamma(X \otimes Y)$ iff $\pi_1[S] \in \Gamma(X)$ and $\pi_2[S] \in \Gamma(Y)$. The linear function space also has $|X \multimap Y| = |X| \times |Y|$, and S is coherent just when it satisfies the conditions given above in the definition of a morphism. (So a clique of $X \multimap Y$ is exactly a morphism from X to Y, as we would expect.) The category is $*$-autonomous with $\perp = I$, so $S \in \Gamma(X^\perp)$ iff $X \notin \Gamma^*(X)$.

Moreover the category has a zero object $\varnothing$ and also sums and products. The sum has $|X + Y| = |X| + |Y|$, and $S \in \Gamma(X + Y)$ iff $S \cap X \in \Gamma(X)$ and $S \cap Y \in \Gamma(Y)$.[5] Similarly $|X \times Y| = |X| + |Y|$ again,

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[5] We are treating the disjoint union using the common informal practice of pretending that $|X|$ is really disjoint from $|Y|$ and we have taken an ordinary union. The pedant's formulation of our $S \cap X$, for example, would be

$$\{x \in X \mid i_1(x) \in S\}$$

and $S \in \Gamma(X \times Y)$ iff

$$S \subseteq X \Rightarrow S \in \Gamma(X)$$

and

$$S \subseteq Y \Rightarrow S \in \Gamma(Y)$$

(so *every* mixed set, i.e. including atoms from both X and Y, is atomically coherent.)

The hypercoherence completion HCoh is a straightforward generalisation of the Hu-Joyal coherence completion. The basic idea is that given a category $\mathbb{C}$, we can construct a new category $\mathrm{HCoh}(\mathbb{C})$ whose objects are hypercoherences in which each atom is tagged with some object of $\mathbb{C}$, and whose morphisms are relations in which each pair is tagged with an appropriate arrow from $\mathbb{C}$.

There is an important subtlety that makes the construction possible in this form. Suppose we have hypercoherences A, B and C with morphisms $f : A \to B$ and $g : B \to C$. If $(a, c) \in gf$ then there is a *unique* $b \in B$ for which $(a, b) \in f$ and $(b, c) \in g$; the reason is that, since f is a hypercoherence morphism and $\{a\}$ (being a singleton) is a clique, the set

$$\{b \in B \mid (a, b) \in f\}$$

must be a clique, and since g is a hypercoherence morphism and $\{c\}$ is an anticlique, the set

$$\{b \in B \mid (b, c) \in g\}$$

must be an anticlique. Thus the intersection of these two sets cannot have more than a single element.

Definition 3.13. Let $\mathbb{C}$ be a category. We define $\mathrm{HCoh}(\mathbb{C})$ as follows. An object is a hypercoherence X together with a function o_X from $|X|$ to the set $|\mathbb{C}|$ of objects of $\mathbb{C}$; we say that the atom $x \in |X|$ is *tagged* with the object $o_X(x)$. A morphism $X \to Y$ is a hypercoherence morphism $f : X \to Y$ together with, for each $(x, y) \in f$, an arrow $a_f(x, y) : o_X(x) \to o_Y(y)$ in $\mathbb{C}$; again we say that the pair (x, y) is tagged with the arrow $a_f(x, y)$. The identity morphism on X consists of all pairs (x, x) for $x \in |X|$, and each such pair is tagged with the identity arrow on $o_X(x)$.

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where i_1 is the left injection map.

If we have morphisms $f : W \to X$ and $g : X \to Y$ then the composite has the underlying hypercoherence morphism gf, and for each $(x, z) \in gf$ we let $a_{gf}(x, z)$ be the composite $a_g(y, z).a_f(x, y)$ where y is the (unique) y for which $(x, y) \in f$ and $(y, z) \in g$.

Proposition 3.14. *The category* $\mathrm{HCoh}(\mathbb{C})$ *has a zero object* $\varnothing$*, and any two objects* X *and* Y *have a sum* $X + Y$ *and a product* $X \times Y$*.*

Proof. Clearly $\varnothing$ is a zero object. The sum and product are defined exactly as in the category of hypercoherences, and for any $x \in X$ and $y \in Y$ we set $o_{X*Y}(x) = o_X(x)$ and $o_{X*Y}(y) = o_Y(y)$, where $*$ is either $+$ or $\times$.

The proof that these definitions have the required universal property mimics exactly the (easy) proof of the corresponding result for hypercoherences. □

More interesting things happen if we have some partial monoidal structure on $\mathbb{C}$.

Proposition 3.15. *If* $\mathbb{C}$ *is partial (symmetric) monoidal then* $\mathrm{HCoh}(\mathbb{C})$ *is (symmetric) monoidal.*

Proof. We define

$$|X \otimes Y| = \big\{(x, y) \in |X| \times |Y| \,\big|\, o_X(x) \otimes o_Y(y) \text{ exists}\big\},$$

and set $o_{X \otimes Y}(x, y) = o_X(x) \otimes o_Y(y)$; the atomic coherence has

$$S \in \Gamma(X \otimes Y) \iff \pi_1[S] \in \Gamma(X) \text{ and } \pi_2[S] \in \Gamma(Y).$$

The unit for our tensor product is I with $|I| = \{i\}$ and $o_I(i)$ the tensor unit in $\mathbb{C}$.

The structural isomorphisms lift directly from $\mathbb{C}$; for example $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ has an underlying hypercoherence morphism consisting of all pairs

$$\big(((a, b), c), (a, (b, c))\big)$$

and each such pair is associated with $\alpha_{o_A(a), o_B(b), o_C(c)}$ in $\mathbb{C}$. □

Proposition 3.16. *If* $\mathbb{C}$ *is partial symmetric monoidal closed then* $\mathrm{HCoh}(\mathbb{C})$ *is symmetric monoidal closed.*

Proof. If $\mathbb{C}$ is closed then we can define $X \multimap Y$ in $\mathrm{HCoh}(X \multimap Y)$ by setting

$$|X \multimap Y| = \{(x, y) \in |X| \times |Y| \mid o_X(x) \multimap o_Y(y) \text{ exists}\}$$

and $S \in \Gamma(X \multimap Y)$ iff

$$\pi_1[S] \in \Gamma(X) \;\Rightarrow\; \pi_2[S] \in \Gamma(Y)$$

and

$$\pi_1[S] \in \Gamma^*(X) \;\Rightarrow\; \pi_2[S] \in \Gamma^*(Y).$$

Unsurprisingly we let $o_{X \multimap Y}(x, y) = o_X(x) \multimap o_Y(y)$. $\square$

Proposition 3.17. *If* $\mathbb{C}$ *is partial* $*$*-autonomous then* $\mathrm{HCoh}(\mathbb{C})$ *is* $*$*-autonomous.*

Proof. We let $\perp$ be the hypercoherence on a singleton $\{b\}$ with $o_\perp(b)$ equal to $\perp$ in $\mathbb{C}$. It follows that $X^\perp$ has the same atoms as X with $S \in \gamma(X^\perp)$ iff $S \notin \Gamma^*(X)$; and $o_{X^\perp}(x) = o_X(x)^\perp$. $\square$

3.3.2 Exponentials

First we recall the standard linear exponential comonads on the category of hypercoherences. There are two well-known such constructions, one based on finite sets of atoms and the other on finite multisets. The simplest formal definition of a finite multiset is:

Definition 3.18. Let X be a set. A *finite multiset* T over X is given by a function $oT : S \to \mathbb{N}$ such that $oT(x) = 0$ for all but finitely many $x \in X$. We write $x \in T$ as an abbreviation for $oT(x) > 0$. The union of two multisets T and T' is $T + T'$ with $o(T + T')(x) = oT(x) + oT'(x)$. Similarly we write $T' \subseteq T$ to mean that $oT'(x) \leq oT(x)$ for every $x \in X$, and say that T' is a *submultiset* of T. The *support* of T is the set $|T| = \{x \in X \mid oT(x) > 0\}$. A multiset T is *non-empty* just when its support is non-empty, i.e. there is some $x \in X$ for which $oT(x) > 0$.

We write $M_f(X)$ for the set of all finite multisets over X. (Note that M_f is a monad on the category of sets, whose algebras are commutative monoids.)

The unqualified word ‘multiset’ below will always mean finite multiset. One can think of a multiset as being a finite set with multiplicity, which we write with square brackets rather than braces. i.e. the multiset $[x, x]$ is not equal to $[x]$ (but of course $[x, y]$ is equal to $[y, x]$). It’s useful to have a notation for multiset comprehensions, so if S is a multiset over X, $f : X \to Y$ is a function, and ϕ is a predicate on X, the multiset

$$T = [f(x) \mid x \in S, \phi(x)]$$

over Y is defined to have

$$oT(y) = \sum \left\{ oS(x) \mid x \in f^{-1}(y), \phi(x) \right\}.$$

We borrow the usual informal conventions of set comprehensions, for example if f is the identity function on X we write $[x \in S \mid \phi(x)]$ for T above. Also we write $f[S]$ as an abbreviation for $[f(x) \mid x \in S]$.

The following auxiliary notion is used in the definition of both hypercoherence exponentials.

Definition 3.19. Let $S \subseteq \mathcal{P}X$ be a family of subsets of some set X. The set $T \subseteq S$ is a *section* of S if

$$(\forall a \in T)(\exists A \in S) a \in A$$

and

$$(\forall A \in S)(\exists a \in T) a \in A.$$

So T is a subset of $\bigcup T$ that intersects each $A \in S$. Alternatively but equivalently, a section of S is a set of the form $\bigcup_{R \in S} T_R$ where for each $R \in S$, T_R is a non-empty subset of R.

Similarly if $S \subseteq M_f X$ is a family of finite multisets over X, a section of S is a multiset of the form $\bigcup_{R \in S} T_R$ where for each $R \in S$, T_R is a non-empty submultiset of R.

Definition 3.20. The *subset exponential* is defined as follows. For each object X, $|!X|$ is the set of finite cliques of X. The atomic coherence uses the notion of section: for any subset S of $|!X|$ we let $S \in \Gamma(!X)$ iff every section of S is coherent in X. Given a morphism $f : X \to Y$ we define $!f : !X \to !Y$ be

$$\{(\pi_1[S], \pi_2[S]) \mid S \subseteq_{\text{fin}} f\}.$$

The functor ! is monoidal with $m_I : I \longrightarrow !I$ equal to $\{(i, \{i\})\}$ and

$$m_{X,Y} = \{((\pi_1[S], \pi_2[S]), S) \mid S \in |!(X \otimes Y)|\}.$$

The counit of the comonad has

$$\varepsilon_X = \{\{x\}, x) \mid x \in |X|\}$$

and the comultiplication has

$$\delta_X = \left\{(\bigcup S, S) \,\middle|\, S \in |!!X|\right\}.$$

The comonoid structure is given by

$$\begin{aligned} e_X &= \{(\varnothing, i)\} \\ d_X &= \{(S \cup T, (S, T)) \mid S \cup T \in |!X|\}. \end{aligned}$$

for every object X.

Definition 3.21. The *multiset exponential* is defined in a similar way to the subset exponential, except that this time $|!X|$ is the set of finite multisets whose support is a clique of X. For example we have

$$!f = [(\pi_1[S], \pi_2[S]) \mid S \in M_f(f)].$$

Of course the set unions $\cup$ are replaced by multiset union $+$ wherever they occur in the definition.

If $\mathbb{C}$ has a linear exponential comonad then we can define linear exponential comonads on HCoh($\mathbb{C}$) corresponding to the two exponential constructions on hypercoherences. In fact the multiset exponential can be constructed using *any* monoidal comonad on $\mathbb{C}$, whether it's an actual linear exponential or even the identity comonad. So the (hyper)coherence completion allows us to embed any $*$-autonomous category into a model for all of linear logic.

To construct the powerset exponential requires a monoidal comonad that supports weakening, so the only part of the linear exponential structure we don't actually need there is the natural transformation e.

As with the monoidal structure, the comonad on $\mathbb{C}$ need only be partially defined.

Definition 3.22. Let $\mathbb{C}$ be a partial monoidal category. A *partial monoidal comonad* on $\mathbb{C}$ is a monoidal comonad $(!, \varepsilon, \delta)$ on $\mathrm{Zero}(\mathbb{C})$, with ! a Z-functor.

A *partial linear exponential comonad* on the partial symmetric monoidal category $\mathbb{C}$ consists of:

- A partial monoidal comonad $(!, \varepsilon, \delta)$;
- with monoidal natural transformations $d_X : !X \to !X \otimes !X$ and $e_X : !X \to I$ on $\mathrm{Zero}(\mathbb{C})$ such that *if* $!X \otimes !X$ *is defined* (*i.e. non-zero*), $(!X, d_X, e_X)$ is a comonoid,[6]
- such that the four diagrams at the end of the statement of Prop. 3.4 commute in $\mathrm{Zero}(\mathbb{C})$.

Proposition 3.23. *If* $\mathbb{C}$ *has a partial monoidal comonad* $(!, \varepsilon, \delta)$*, there is a corresponding linear exponential comonad on* $\mathrm{HCoh}(\mathbb{C})$*, based on the multiset exponential.*

Proof. For the purposes of this proof we shall omit the associativity any symmetry isomorphisms of $\mathbb{C}$, pretending that the tensor product is commutative and associative on the nose. This is admissible provided we are careful, since (by the coherence theorem) every monoidal category is equivalent to a strict monoidal category, and in a symmetric strict monoidal category there is exactly one structural isomorphism corresponding to each permutation of a finite list. As part of this temporary convention, we allow ourselves to write expressions like $\bigotimes_{x \in S} f(x)$, where S is a multiset and $f(x)$ is an expression possibly containing x. The value of such an expression is unique up to the permutation isomorphisms, so when such an expression is used in a definition we rely on the axiom of choice to select a particular object from each isomorphism class. Then an arrow between two such expressions is implicitly composed with the permutation isomorphism that matches up the elements appropriately.

As a result of this convention, for any multiset S and function $f : |S| \to |\mathbb{C}|$ we can write

$$m_S^f : \bigotimes_{x \in S} !f(x) \to ! \bigotimes_{x \in S} f(x)$$

---

[6] One might imagine that this condition is vacuous, since if $X \otimes X$ is a zero object we always have a zero map $X \to X \otimes X$. However this can never satisfy the unit conditions in the definition of a comonoid.

for the monoidality transformation of the functor !.

For any object X of $\mathrm{HCoh}(\mathbb{C})$ we let $|!X|$ be

$$\Big\{S \in M_f(|X|) \;\Big|\; (\forall T \subseteq |S|)[T \in \Gamma(X)] \text{ and } \bigotimes_{x \in S} !o_X(x) \text{ exists}\Big\},$$

and let $o_{!X}(S) = \bigotimes_{x \in S} !o_X(x)$.

A multiset A over $|!X|$ is coherent in $!X$ just when all its sections are coherent in X, exactly as in the ordinary category of hypercoherences.

The effect of ! on morphisms is as follows. If we have a morphism $f : X \to Y$ in $\mathrm{HCoh}(\mathbb{C})$ then $!f$ consists of all pairs $(\pi_1[S], \pi_2[S])$ such that S is a finite multiset over f. Each such pair is tagged with the arrow $\bigotimes_{(x,y) \in S} !a_f(x, y)$. It's clear then that this is a functor.

The counit of the comonad has components $\varepsilon_X : !X \to X$ consisting of all pairs $([x], x)$ for $[x] \in |!X|$;[7] each such pair is tagged with the arrow $\varepsilon_{o_X(x)} : !o_X(x) \to o_X(x)$ in $\mathbb{C}$, which has the correct type since $o_{!X}([x]) = !o_X x$. The comonad comultiplication has components $\delta_X : !X \to !!X$ consisting of all pairs $(\sum S, S)$ for $S \in |!!X|$. To understand how such a pair should be tagged, it's useful to consider a simple example. Suppose $S = [[a, a'], [b, b']]$, and to keep the notation manageable abbreviate $o_X(x)$ as x for each $x \in |X|$. Then the pair $(\sum S, S)$ has to be tagged with an arrow of type

$$!a \otimes !a' \otimes !b \otimes !b' \to !(!a \otimes !a') \otimes !(!b \otimes !b').$$

For this we take the tensor product of

$$!a \otimes !a' \xrightarrow{\delta_a \otimes \delta_{a'}} !!a \otimes !!a' \xrightarrow{m_{!a,!a'}} !(!a \otimes !a')$$

with

$$!b \otimes !b' \xrightarrow{\delta_b \otimes \delta_{b'}} !!b \otimes !!b' \xrightarrow{m_{!b,!b'}} !(!b \otimes !b').$$

In general, then, the arrow that tags a pair $(\sum S, S)$ is

$$\bigotimes_{T \in S} \Big(m^!_T \bigotimes_{x \in T} \delta_{o_X(x)} \Big),$$

where $m^!$ is an abbreviation for $m^{\lambda x. !o_X(x)}$.

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[7] Notice the subtle difference here with the definition given for plain hypercoherences. The point is that $!o_X(x)$ might not exist, so it's not necessarily the case that $[x] \in |!X|$ for every $x \in |X|$.

The comonoid structure is thankfully fairly simple. For the counit $e_X : !X \to I$ we have the hypercoherence morphism $\{([], \iota)\}$ whose element is tagged with the identity on $I \in \mathbb{C}$. The comultiplication arrow $d_X : !X \to !X \otimes !X$ has the underlying hypercoherence morphism $\{(S+T, (S, T)) \mid S, T \in |!X|\}$ and each pair $(S+T, (S,T))$ is tagged with $\bigotimes_{x \in S+T} 1_{o_X(x)}$. The proof that $(!X, e_X, d_X)$ is a comonoid for every $X \in \mathrm{HCoh}(\mathbb{C})$ mimics exactly the corresponding proof for plain hypercoherences, since the tags are all identity arrows.

Our next task is to check that $(!, \varepsilon, \delta)$ is indeed a comonad, which is unfortunately not so easy. Let X be any object of $\mathrm{HCoh}(\mathbb{C})$; we must show first that the composites $\varepsilon_{!X}\delta_X$ and $!(\varepsilon_X)\delta_X$ are both the identity on $!X$. Take any $S \in |!X|$; we'll show that in either composite it's related only to itself, tagged with the identity arrow. First consider $\varepsilon_{!X}$; here each $S \in |!X|$ is related only to $\{S\} \in |!!X|$, tagged with $\varepsilon_{o_{!X}(S)}$ in $\mathbb{C}$. Each such singleton $\{S\}$ relates (by δ) only to its element S, and the pair $(S, \{S\})$ is tagged with $m^!_S \bigotimes_{x \in S} \delta_x$. We must show that the composite of these arrows in $\mathbb{C}$ is the identity: if we simply write $\otimes$ for $\bigotimes_{x \in S}$, our composite is the top and right in

$$\begin{tikzcd}
\otimes x \arrow[r, "\otimes \delta_x"] \arrow[dr, "1"'] & \otimes !!x \arrow[r, "m^!_S"] \arrow[d, "\otimes \varepsilon_{!x}"] & !(\otimes !x) \arrow[dl, "\varepsilon_{!x \otimes !y}"] \\
& \otimes !x &
\end{tikzcd}$$

where the right-hand triangle commutes because ε is a monoidal natural transformation in $\mathbb{C}$ and the left-hand triangle commutes because $(!, \varepsilon, \delta)$ is a comonad in $\mathbb{C}$.

For the $!\varepsilon_X$ case, each multiset $S = [x, y, \ldots]$ in $!X$ is related (by $!\varepsilon_X$) only to the multiset $[[x], [y], \ldots]$ of singletons in $!!X$, tagged with the arrow $\bigotimes_{x \in S} !(\varepsilon_x)$. Each such set of singletons is related (by δ) only to S in $!X$, tagged with $\bigotimes_{x \in S} \delta_x$. This time it's immediate that the tags compose to give the identity, since $(!, \varepsilon, \delta)$ is a comonad in $\mathbb{C}$.

The remaining diagram whose commutativity must be established is the following.

$$\begin{tikzcd}
!X \arrow[r, "\delta_X"] \arrow[d, "\delta_X"'] & !!X \arrow[d, "\delta_{!X}"] \\
!!X \arrow[r, "!\delta_X"'] & !!!X
\end{tikzcd}$$

Suppose we have some element S in $!!!X$, which is tagged with

$$\bigotimes_{R\in S} ! \bigotimes_{Q\in R} ! \bigotimes_{x\in Q} !o_X(x).$$

There's a unique element $\sum S \in\ !!X$ related by $\delta_{!X}$ to S, and tagged with

$$\bigotimes_{R\in S} \bigotimes_{Q\in R} ! \bigotimes_{x\in Q} !o_X(x);$$

similarly there's a unique element $\{\sum R \mid R \in S\}$ related to S by $!\delta_X$, tagged with

$$\bigotimes_{R\in S} ! \bigotimes_{Q\in R} \bigotimes_{x\in Q} !o_X(x).$$

Then if we omit the subscripts for the sake of clarity, we have the following diagram:

$$\begin{tikzcd}
\otimes\otimes\otimes !x \arrow[r, "\otimes\otimes\otimes\delta_x"] \arrow[d, "\otimes\otimes\otimes\delta_x"'] & \otimes\otimes\otimes !!x \arrow[r, "\otimes\otimes m^{!x}"] \arrow[d, "\otimes\otimes\otimes\delta_{!x}"] & \otimes\otimes !(\otimes !x) \arrow[dd, "\otimes\otimes\delta_{\otimes !x}"] \\
\otimes\otimes\otimes !!x \arrow[r, "\otimes\otimes\otimes !\delta_x"'] \arrow[d, "\otimes\otimes m^{!x}"'] & \otimes\otimes\otimes !!!x \arrow[d, "\otimes\otimes m^{!!x}"] & \\
\otimes\otimes !(\otimes !x) \arrow[r, "\otimes\otimes !\otimes\delta_x"'] \arrow[d, "\otimes m^{\otimes !x}"'] & \otimes\otimes !(\otimes !!x) \arrow[r, "\otimes\otimes !m^{\otimes !x}"'] \arrow[d, "\otimes m^{\otimes !!x}"] & \otimes\otimes !!(\otimes !x) \arrow[d, "\otimes m^{!(\otimes !x)}"] \\
\otimes !(\otimes\otimes !x) \arrow[r, "\otimes !(\otimes\otimes\delta_x)"'] & \otimes !(\otimes\otimes !!x) \arrow[r, "\otimes !(\otimes m^{!x})"'] & \otimes !(\otimes !(\otimes !x))
\end{tikzcd}$$

The upper left cell commutes because $(!, \varepsilon, \delta)$ is a comonad in $\mathbb{C}$, the middle and lower left and the lower right because m is natural, and the upper right because δ is monoidal. It follows that the required diagram for δ does indeed commute.

The monoidality of ! is essentially as for plain hypercoherences. We have $m_I : I \to\ !I$ equal to $\{(i, [i])\}$ with $a_{m_I}(i, [i]) = m_I$, and

$$m_{X,Y} = \{((\pi_1[S], \pi_2[S]), S) \mid S \in |!(X \otimes Y)|\}.$$

and each pair $((\pi_1[S], \pi_2[S]), S)$ is tagged with

$$\bigotimes_{(x,y)\in S} m_{o_X(x), o_Y(y)}.$$

The naturality of m and the commutativity of the seven diagrams shown in the statement of Prop. 3.4 succumb to routine calculation. We conclude therefore that we have indeed defined a linear exponential comonad in $\mathrm{HCoh}(\mathbb{C})$. ☐

Proposition 3.24. *If* $\mathbb{C}$ *has a partial linear exponential comonad, there is a corresponding linear exponential comonad on* $\mathrm{HCoh}(\mathbb{C})$*, based on the subset exponential.*

Proof sketch. The difference here is that we sometimes need to duplicate elements when we construct the tags, for which we have to use the contraction transformation d. For example, if we have $\{\{a, b\}, \{b, c\}\} \in |!!X|$, the pair

$$\big(\{a, b, c\}, \{\{a, b\}, \{b, c\}\}\big) \in \delta_X$$

needs to be tagged with the composite

$$\begin{aligned} !a \otimes !b \otimes !c &\xrightarrow{!a \otimes d_b \otimes !c} !a \otimes !b \otimes !b \otimes !c \\ &\xrightarrow{\delta_a \otimes \delta_b \otimes \delta_b \otimes \delta_c} !!a \otimes !!b \otimes !!b \otimes !!c \\ &\xrightarrow{m_{!a,!b} \otimes m_{!b,!c}} !(!a \otimes !b) \otimes !(!b \otimes !c). \end{aligned}$$

The tags of the d transformation in $\mathrm{HCoh}(\mathbb{C})$ also need similar treatment. ☐

3.3.3 Exponential on $\mathbb{C} + \mathbb{C}^{op}$

Now we can show that under reasonable conditions it's possible to define a partial linear exponential on $\mathbb{C} + \mathbb{C}^{op}$.

Definition 3.25. Suppose we have a linear exponential comonad $(!, \varepsilon, \delta)$ on the category $\mathbb{C}$. A *co-exponential monad* on $\mathbb{C}$, compatible with !, consists of a monad (M, η, μ) on $\mathbb{C}$ and a natural transformation β (the *compatibility*) having components

$$\beta_{X,Y} : M(X \multimap Y) \to\, !X \multimap MY$$

such that for every two elements X and Y, the following two diagrams commute:

$$\begin{tikzcd}
M(X \multimap Y) \arrow[rr, "\beta_{X,Y}"] & & !X \multimap MY \\
MM(X \multimap Y) \arrow[u, "\mu_{X \multimap Y}"] \arrow[r, "M\beta_{X,Y}"'] & M(!X \multimap MY) \arrow[r, "\beta_{!X,MY}"'] & !!X \multimap MMY \arrow[u, "\delta_X \multimap \mu_Y"']
\end{tikzcd} \tag{3.8}$$

$$\begin{tikzcd}
X \multimap Y \arrow[r, "\mathrm{id}"] \arrow[d, "\eta_{X \multimap Y}"'] & X \multimap Y \arrow[d, "\varepsilon_X \multimap \eta_Y"] \\
M(X \multimap Y) \arrow[r, "\beta_{X,Y}"'] & !X \multimap MY
\end{tikzcd} \tag{3.9}$$

Example 3.26. If $\mathbb{C}$ has a zero object, we can take M to be the constant-zero functor and all the components of β to be zero maps. Thus if a category $\mathbb{C}$ has a linear exponential monad then Zero($\mathbb{C}$) necessarily has a linear exponential comonad and a trivial compatible co-exponential monad. (Of course there may also be other more interesting co-exponential structures.)

Example 3.27. In the category of simple games the *Lamarche-Curien exponential* can be informally described by saying that !X is obtained from X by allowing Opponent to 'backtrack' at any time to an earlier unused position. The dual notion, a game MX in which Proponent is allowed to backtrack, gives rise to a co-exponential monad compatible with !.

Proposition 3.28. *If we have a linear exponential comonad* $(!, \varepsilon, \delta)$ *and a compatible co-exponential monad* (M, η, μ) *on* Zero($\mathbb{C}$)*, then we can construct a partial linear exponential* ‼ *on* Zero($\mathbb{C} + \mathbb{C}^{op}$).

Proof. Let $‼X^+ = (!X)^+$ and $‼X^- = (MX)^-$. Clearly this is a comonad with

$$\begin{aligned} \varepsilon_{X^+} &= (\varepsilon_X)^+ & \delta_{X^+} &= (\delta_X)^+ \\ \varepsilon_{X^-} &= (\eta_X)^- & \delta_{X^-} &= (\mu_X)^-. \end{aligned}$$

The monoidal structure is given by $m_{I^+} = (m_I)^+$ and

$$\begin{aligned} m_{X^+,Y^+} &= (m_{X,Y})^+ \\ m_{X^+,Y^-} &= (\beta_{X,Y})^- \\ m_{X^-,Y^+} &= (\beta_{Y,X})^- \\ m_{X^-,Y^-} &= 1_Z. \end{aligned}$$

Our conditions on β ensure that this m is natural. The tensor product $!X^\pm \otimes !X^\pm$ exists only if $X^\pm$ is positive, so we just set $d_{X^+} = (d_X)^+$ and $e_{X^+} = (e_X)^+$. $\square$

3.3.4 Hypergraphs

The terminal category $\mathbf{1}$ has a trivial $*$-autonomous structure, and a linear exponential comonad given by the identity functor, with a compatible co-exponential monad also given by the identity functor. It follows that $\mathbf{1}+\mathbf{1}^{\mathrm{op}}$ has a canonical partial $*$-autonomous structure using the construction above. The category $\mathbf{1}+\mathbf{1}^{\mathrm{op}}$ has two objects, I^+ and I^-, and only identity morphisms.

An object of $\mathrm{HCoh}(\mathbf{1}+\mathbf{1}^{\mathrm{op}})$ is thus a hypercoherence in which each atom is tagged either with I^+, in which case we call it *positive*, or with I^-, when we call it *negative*. It's now easy to check that the morphisms and operations on this category correspond exactly to the usual definitions for hypergraphs, as given by Melliès (2003, §3).

Chapter 4

The Future

This chapter describes work that's currently in progress, and outlines our plans for the rest of the project.

4.1 Games as Free Structures

Cockett and Seely (2004) have recently discovered that a certain simple category of games is the initial (i.e. completely free) model of a certain categorical situation. Because of their focus on 'polarized categories' they present the result in a way that involves two categories; here we describe a one-sided version of the theory. The Cockett-Seely result concerns the category of finitary simple games with winning strategies.

Definition 4.1. A *finitary* simple game is a simple game in which the (prefix-ordered) poset of plays is bounded.

The situation in question is this:

Definition 4.2. A *G-category* is a category $\mathbb{C}$ with all small (i.e. set-indexed) products that is equipped with a contravariant endofunctor $\uparrow$ that is self-adjoint on the right in the sense that $\mathbb{C}(X, \uparrow Y)$ is naturally isomorphic to $\mathbb{C}(Y, \uparrow X)$.

A *G-functor* between G-categories $\mathbb{C}$ and $\mathbb{D}$ is a product-preserving functor $F : \mathbb{C} \to \mathbb{D}$ such that $F(\uparrow X) = \uparrow F(X)$ for any object or morphism X in $\mathbb{C}$.

The unit (and counit) of the adjunction is an arrow $\chi_X : X \to \uparrow\uparrow X$ such that

$$\uparrow X \xrightarrow{\chi_{\uparrow X}} \uparrow\uparrow\uparrow X \xrightarrow{\uparrow \chi_X} \uparrow X$$

is the identity.

For the purposes of this section we treat a morphism $G \to H$ as a strategy on $G \multimap H$.

In the category of finitary simple games, the $\uparrow$ functor is the 'lifting' functor defined as

$$\begin{aligned}(\uparrow G)_O &= \{*_G\} + G_P \\ (\uparrow G)_P &= G_O \\ \mathrm{pos}(\uparrow G) &= \{\varepsilon\} \cup \{*_G p \mid p \in \mathrm{pos}(G)\},\end{aligned}$$

and given $f : G \to H$ we let $\uparrow f : \uparrow H \to \uparrow G$ be

$$\{*_G *_H p \mid p \in f\}.$$

Notice that a morphism $G \to \uparrow H$ is given by a prefix-closed subset of

$$(G_O G_P + H_O H_P)^*$$

whose projections onto G and H are valid positions. Since this is symmetrical in G and H, it's clear that the lifting functor must be self-adjoint on the right.

Observe also that, up to isomorphisms given by relabelling moves, every finitary simple game may be constructed from the trivial game (which is the terminal object) using lifting and (possibly infinite) products. Cockett and Seely's result can be stated as follows.

Proposition 4.3 (Cockett and Seely (2004)). *Up to equivalence, the category of finitary simple games is the initial G-category.*

The point is that – as we have described – the category of games is a G-category, and every strategy $G \to H$ can be described by a unique combination of the structural maps that exist in any G-category.

There are also descriptions of this sort for other categories of (finitary) games. For example, say that a category has *one-sided zero morphisms* if for every two objects X and Y there is a distinguished map $z_{X,Y}$, and for any $f : W \to X$ we have $z_{X,Y} f = z_{W,Y}$. The category of simple games (where the strategies are not necessarily winning) has one-sided zero morphisms given by the trivial strategy containing only the initial position ε. Note that these are not actual (two-sided) zero morphisms. Now say that a G'-category is a G-category in which every homset has a one-sided zero morphism. Then the category of finitary simple games (whose strategies need not be winning) is the free G'-category.

We believe that it's also possible to characterise the category of games with innocent strategies (Hyland and Ong, 2000) as the initial model of a slightly more complicated situation. *A tensor-lift category* is a symmetric monoidal category $\mathbb{C}$ with a contravariant endofunctor $\uparrow$ such that for any object Y, the functor $\lambda X.\uparrow(X \otimes Y)$ is self-adjoint on the right. In particular any $*$-autonomous category is a tensor-lift category, with the lift given by negation. However the converse does not hold, since the unit of our adjunction need not be an isomorphism.

Conjecture 4.4. The category of arena games with innocent, winning strategies is the free tensor-lift category with products in which the tensor unit is a terminal object.

There is an intriguing similarity between the notion of tensor-lift category and the weakened definition of *response category* described by Selinger (2003). Indeed, a response category is exactly a tensor-lift category whose monoidal structure is cartesian (i.e. I is a terminal object and $\otimes$ is a categorical product.) The 'response object' is $\uparrow I$, and we have

$$\hom(B, \uparrow A) \cong \hom(B \otimes A, \uparrow I)$$

as required. It seems likely that the free response category can be described as a category of games. Also it follows from Selinger's results that the free response category must be a control category, hence a model for the call-by-name $\lambda\mu$-calculus. We plan to study this category and examine whether it gives an interesting model of control.

Another potentially interesting object of study is the free $*$-autonomous tensor-lift category (with the lift independent of the negation). Preliminary investigations suggest that this gives a new kind of game category that is apparently well-suited for building accurate models of linear logic. We hope that this line of inquiry will allow the fully abstract model of MALL to be presented as a category of games.

4.2 Categorical Construction of Innocence

Hyland's (2004) observation is that the (cartesian closed) category of games with innocent strategies can be constructed from the category of simple games as a particular biKleisli category. The description below is reconstructed from notes taken at Hyland's talk, and does not contain all the details, though nothing important seems to be missing. Understanding the details is on the agenda for future work.

The general setting for the construction is a symmetric monoidal closed category $\mathbb{G}$ with finite products, equipped with a linear exponential comonad $(!, \varepsilon, \delta)$ and a monad $(?, \eta, \mu)$ such that $?$ preserves finite products. There is also a bifunctor

$$-\!\boxplus : \mathbb{G}^{\mathrm{op}} \times \mathbb{G} \to \mathbb{G}$$

with the property that

$$?(A -\!\boxplus B) \cong {!A} \multimap {?B}$$

naturally in both arguments, and a distributive law $!? \xrightarrow{\lambda} ?!$ (see Barr and Wells, 1985, §9.2), which means that there is a *biKleisli category* for $?$ and $!$ in which an object is an object of $\mathbb{G}$ and an arrow $A \to B$ is given by an arrow

$$!A \to ?B$$

from $\mathbb{G}$. We compose $!A \xrightarrow{f} ?B$ with $!B \xrightarrow{g} C$ as follows:

$$!A \xrightarrow{\delta} !!A \xrightarrow{!f} !?B \xrightarrow{\lambda} ?!B \xrightarrow{?g} ??C \xrightarrow{\mu} ?C,$$

and the identity is

$$!A \xrightarrow{\varepsilon} A \xrightarrow{\eta} ?A.$$

Because $?$ preserves products, the product on $\mathbb{G}$ is also a product on the biKleisli category. Furthermore the biKleisli category is cartesian closed with the function space given by $\mathrel{-\!\boxplus}$, since we have

$$\begin{aligned} & \mathbb{G}(!A, ?(B \mathrel{-\!\boxplus} C)) \\ \cong\ & \mathbb{G}(!A, !B \multimap ?C) \\ \cong\ & \mathbb{G}(!A \otimes !B, ?C) \\ \cong\ & \mathbb{G}(!(A \times B), ?C) \end{aligned}$$

naturally.

4.3 Glueing and Orthogonality

The techniques of *glueing and orthogonality* (Hyland and Schalk, 2003) give a general framework for constructing complicated $*$-autonomous categories from simpler ones. Very recently, in collaboration with Richard Garner in Cambridge, we have discovered a new way to describe the category of hypercoherences using these techniques. Our construction establishes a precise sense in which the structure of HCoh is lifted from the structure of Rel.

Furthermore (while this chapter was being written) we have apparently found a way to describe the coherence completion using a similar technique. There is good reason to suppose that a suitable combination of these two constructions should yield an abstract description of the hypercoherence completion.

This work is too new to describe in detail here – indeed, not all of the details have yet been worked out precisely – and will form a basis for one strand of future research. As a result of these recent discoveries, the timetable below differs slightly from the one given in the short end of year report.

4.4 Timetable

A rough timetable for the remainder of the project is below. It is structured so as to favour those questions which seem most likely to have an impact on the direction of subsequent research.

Oct–Nov 2004	Describe the free $\otimes\times\uparrow$ category, and establish its relationship with innocence. (What about well-bracketing?)
Dec–Jan 2005	Use the techniques of Glueing and Orthogonality to study completion processes (if possible)
Feb–Mar 2005	Describe the free response category, and establish how it relates to known categories of games. Study control categories and the $\lambda\mu$-calculus. Investigate applications.
Apr–May 2005	Describe games with involutive negation. Give a proof-theoretic description of this category using sequent calculus. Can this be used to build a fully-complete model of MALL or MLLU?
Jul–Jul 2005	Study Hyland's *Categorical Construction of Innocence*, and relate it to the work above.
Aug–Sep 2005	Investigate the higher-dimensional algebra underlying earlier study of completion processes.
Oct–Feb 2006	Study the issues arising from the work so far.
Mar–Aug 2006	Write it all up!

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